Tomasito: "Transport and disorder"

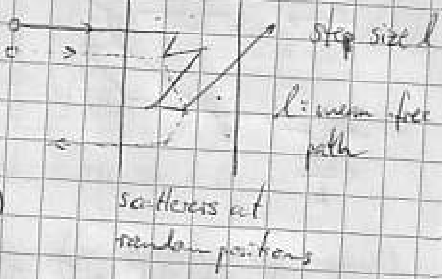
1) Introduction

1.1. classical particel in random potential

ensemble average

→ diffusion eq.

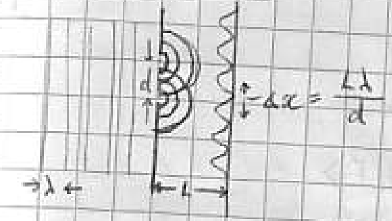
$$\frac{\partial}{\partial t}\, p(\vec{r},t) - D\Delta p(\vec{r},t) = j(\vec{r},t)$$



with: $D = \frac{vl}{3}$ (in 3D) v: velocity of particle

1.2. waves and interference

1.2.1. double slit



path 1 path 2

$$\psi = \psi_1 + \psi_2 \quad \in \mathbb{C}$$

$$\Rightarrow |\psi|^2 = |\psi_1|^2 + |\psi_2|^2 + \underbrace{2\,\mathrm{Re}\;\psi_1\psi_2^*}_{\text{interference term}}$$

Possible reasons for vanishing interferences:

- incoherent source
- quantum-mechanical decoherence ("collapse")
  "which way information"
- disorder: positions of slits fluctuates

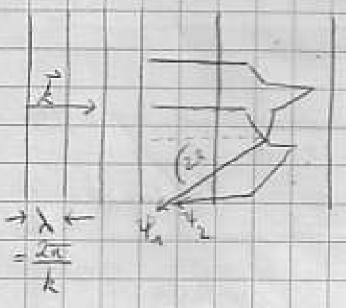
$$\rightsquigarrow \langle|\psi|^2\rangle = \langle|\psi_1|^2\rangle + \langle|\psi_2|^2\rangle + 2\,\mathrm{Re}\,\underset{\downarrow}{\cancel{\langle\psi_1\psi_2^*\rangle}}$$

$$\downarrow$$

$$\psi_1\psi_2^* = |\psi_1||\psi_2| \, e^{i(\varphi_1 - \varphi_2)}$$

$\rightarrow 0$ if φ_1, φ_2 fluctuate

1.2.2. disordered medium



$$\psi = \sum_n \psi_n = \sum_n |\psi_n| \, e^{ikL_n}$$

, with L_n length of scattering path

intensity $J = |\psi|^2 = \sum_{m,n} \psi_m \psi_n^* = \sum_n |\psi_n|^2 + \sum_{m \neq n} \psi_m \psi_n^*$

Interferences lead to speckle pattern

- 1st approximation:

ensemble / disorder - average

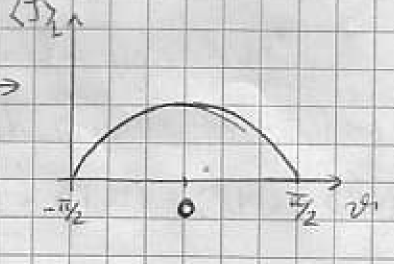
$\hookrightarrow$ interferences vanish

$$\langle J \rangle \approx \underset{\substack{\uparrow \\ \text{ladder diagrams}}}{\langle J \rangle_L} = \sum_n \langle |\psi_n|^2 \rangle$$

Sum over classical trajectories

$\rightarrow$ (random walk / diffusion equation)

$\longrightarrow$

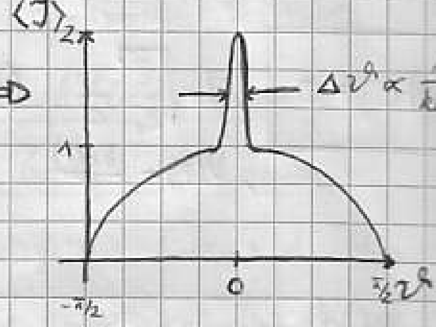


Do paths exist $\psi_n \neq \psi_m$ with

even if you change

Yes! reversed scattering

$\Rightarrow$



⇨ There are inter

disorder averag

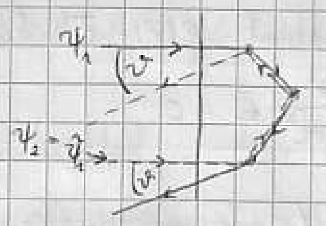
other examples: - weak loca

- strong / And

→ co

Do paths exist $\psi_n \neq \psi_m$ with constant phase difference even if you change the positions of the scatterers?

Yes! reversed scattering paths:

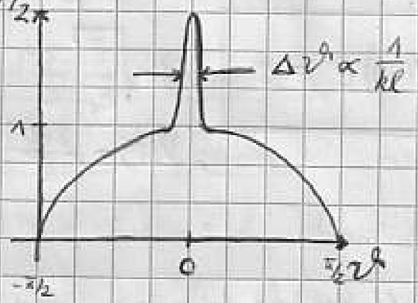


$\vartheta = 0$: for every realization of the medium $L_1 = L_2$

$\Rightarrow \psi_1 = \psi_2$

(reciprocity symmetry)

$\Rightarrow$



coherent back scattering (CBS)

$\Rightarrow$ There are interferences which survive disorder average!

other examples: - weak localization: $\frac{\delta D}{D} \propto \frac{1}{(kl)^2}$ (3D)

- strong/Anderson localization: $D = 0$ for $kl \lesssim 1$

→ complete supression of diffusion

1.3. Wave equations and disordered media

1.3.1. Electromagnetic waves (Helmholtz - equation)

Maxwell's equations in dielectric medium

with dielectric constant $\varepsilon(\vec r) = \bar\varepsilon + \delta\varepsilon(\vec r)$

$$\vec\nabla \times \vec\nabla \times \vec E + \frac{\varepsilon(\vec r)}{\varepsilon_0} \frac{\partial^2}{c^2 \partial t^2} \vec E = 0$$

replace $\vec E$ by E (scalar approximation)

$$\longrightarrow \Delta E + \frac{\varepsilon(\vec r)}{\varepsilon_0} \frac{\partial^2}{c^2 \partial t^2} E = 0$$

stationary solutions

$$E(\vec r, t) = e^{i\omega t} E(\vec r) \leftrightarrow \boxed{\Delta E + k^2 (1 + V(\vec r)) E = 0} \quad , k = \sqrt{\frac{\bar\varepsilon}{\varepsilon_0}} \frac{\omega}{c} \quad (1.1.)$$

spatial part

$$, V(\vec r) = \frac{\delta\varepsilon(\vec r)}{\bar\varepsilon}$$

1.3.2. Electron in disordered solids

1-particle Hamiltonian : $H = -\frac{\hbar^2}{2m} \Delta + V(\vec r)$

Stationary Schr. eq. :

$$\boxed{(\Delta + k^2 (1 + \tilde V(\vec r))) \psi = 0} \quad , k^2 = \frac{2m\omega}{\hbar} \quad (1.2.)$$

$$, \tilde V(\vec r) = -\frac{V(\vec r)}{\hbar\omega}$$

equivalent to equation (1.1.).

1.3.3. Disorder model : locali

$$V(\vec r) = \sum_{i=1}^{N} V_0(\vec r - \vec r_i) \quad \text{generously} \quad \vec r_1, \dots, \vec r_N$$

V_0 potential of single scatte

We are interested in the a

$\langle \psi(\vec r) \rangle$ average

$\langle |\psi(\vec r)|^2 \rangle$ average in

2) Scattering theory

2.1. Green's function

Using the Green's function

the wave equation (1.2.)

terms: $\Delta \psi(\vec r) + k^2 (1 + V(\vec r$

Definition: $\Delta_r G(\vec r, \vec r') + k^2($

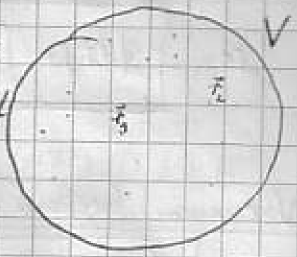
There are 2 solutions

correspond to incou

if $V(\vec r) \to 0$ for $|\vec r$

1.3.3. Disorder model: localized scatterers

$$V(\vec{r}) = \sum_{i=1}^{N} V_0(\vec{r}-\vec{r}_i)$$ $\vec{r}_1, \dots, \vec{r}_N$ homogenously distributed in V



V_0 potential of single scatterer

We are interested in the average quantities

$\langle \psi(\vec{r}) \rangle$ average amplitude

$\langle |\psi(\vec{r})|^2 \rangle$ average intensity

2) Scattering theory

2.1. Green's function

Using the Green's function $G(\vec{r},\vec{r}')$ we can solve the wave equation (1.2.) for arbitrary source terms:

$$\boxed{\Delta \psi(\vec{r}) + k^2 (1+V(\vec{r}))\psi(\vec{r}) = j(\vec{r})} \qquad (2.1.)$$

Definition: $$\boxed{\Delta_r G(\vec{r},\vec{r}') + k^2(1+V(\vec{r}))G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')} \qquad (2.2.)$$

There are 2 solutions (G and G^*)

correspond to incoming or outgoing waves

if $V(\vec{r}) \to 0$ for $|\vec{r}| \to \infty$: $$\begin{cases} G(\vec{r},\vec{r}') \longrightarrow \frac{e^{ikr}}{r} \\ G^*(\vec{r},\vec{r}') \longrightarrow \frac{e^{-ikr}}{r} \end{cases}$$

Solution of (2.1.) (with outgoing boundary cond.)

is $\boxed{\psi(\vec{r}) = \int d\vec{r}\,' \, G(\vec{r}, \vec{r}\,') \, j(\vec{r}\,')}$ (2.3.)

<u>Vacuum Green's function</u> Eq. (2.2.) for $V \equiv 0$

$$\boxed{\left(\Delta_r + k^2\right) G_0(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,')} \quad (2.4.)$$

Solution in 3D: $\boxed{G_0(\vec{r}, \vec{r}\,') = -\frac{e^{ik|\vec{r} - \vec{r}\,'|}}{4\pi |\vec{r} - \vec{r}\,'|}}$ (2.5.)

outgoing spherical wave

2.2. Born-series (Lippman-Schwinger-eq.)

We rewrite eq. (2.1) as an integral equation

$$\boxed{\psi(\vec{r}) = \psi_0(\vec{r}) - k^2 \int d\vec{r}\,' \, G_0(\vec{r}, \vec{r}\,') \, V(\vec{r}\,') \, \psi(\vec{r}\,')} \quad (2.6.)$$

, where ψ_0 is the incoming wave (solution to the vacuum wave equation)

e.g. $\psi_0(\vec{r}) = e^{i \vec{k}_{in} \cdot \vec{r}}$
(for $j = 0$)

Formal solution of eq. (

$$\psi(\vec{r}) = \psi_0(\vec{r}) - k^2 \int d\vec{r}\,' \, G_0(\vec{r},$$

$$+ k^4 \int d\vec{r}\,' d\vec{r}\,'' \, G_0($$

$$- \ldots$$

$\longrightarrow$ diagrams $\longrightarrow$

Born-approximation $\psi =$

... is valid only, if

the medium is

Formal solution of eq. (2.6.) by iteration:

$$\psi(\vec{r}) = \psi_0(\vec{r}) - k^2 \int d\vec{r}\,' \, G_0(\vec{r},\vec{r}\,') V(\vec{r}\,') \psi_0(\vec{r}\,') \qquad (2.7.)$$

$$+ k^4 \int d\vec{r}\,' d\vec{r}\,'' \, G_0(\vec{r},\vec{r}\,') V(\vec{r}\,') G_0(\vec{r}\,',\vec{r}\,'') V(\vec{r}\,'') \psi_0(\vec{r}\,'')$$

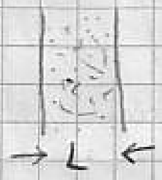
$$- \cdots$$

$\longrightarrow$ diagrams $$\left.\begin{array}{ll} \longrightarrow & G_0 \\ \cdot & V \end{array}\right\}$$

multiple scattering

Born-approximation $\psi = \psi_0 - k^2 G_0 V \psi_0$ (only single scattering)

... is valid only, if the optical thickness of the medium is small.



opt. thickness $b = \frac{L}{\ell}$

Tanasita:

2.3. T-matrix

Def.: $$\boxed{\psi(\vec r) = \psi_0(\vec r) + \int d\vec r\,' d\vec r\,'' \underset{\substack{\uparrow \\ \text{outgoing wave}}}{G_0(\vec r, \vec r\,')}\, \underset{\substack{\uparrow \\ \text{scattering}}}{t(\vec r\,', \vec r\,'')}\, \underset{\substack{\uparrow \\ \text{incoming wave}}}{\psi_0(\vec r\,'')}} \quad (2.8.)$$

According to the Born-series (2.7.) we obtain:

$$\begin{aligned} t(\vec r, \vec r\,') &= -k^2 V(\vec r)\,\delta(\vec r - \vec r\,') \\ &\quad + k^4 V(\vec r)\, G_0(\vec r, \vec r\,')\, V(\vec r\,') \\ &\quad - \dots \end{aligned} \qquad (2.9.)$$

momentum-space-representation

$$\tilde t(\vec k_1, \vec k_2) = \frac{1}{(2\pi)^3} \int d\vec r_1\, d\vec r_2\; e^{-i\vec k_1\cdot\vec r_1 + i\vec k_2\cdot\vec r_2}\; t(\vec r_1, \vec r_2) \quad (2.10)$$

$$t(\vec r_1, \vec r_2) = \frac{1}{(2\pi)^3} \int d\vec k_1\, d\vec k_2\; e^{i\vec k_1\cdot\vec r_1 - i\vec k_2\cdot\vec r_2}\; \tilde t(\vec k_1, \vec k_2) \quad (2.11)$$

Inserting into eq (2.9) yields (for $\psi_0 = e^{i\vec k_{in}\cdot\vec r}$):

$$\psi(\vec r) = \psi_0(\vec r) + \int d\vec r_1\, d\vec r_2\; G_0(\vec r, \vec r_1) \left(\frac{1}{2\pi}\right)^3 \int d\vec k_1\, d\vec k_2\; e^{-i\vec k_1\cdot\vec r_1 + i\vec k_2\cdot\vec r_2}\; \tilde t(\vec k_1, \vec k_2)\, e^{i\vec k_{in}\cdot\vec r_2}$$

$$= e^{i\vec k_{in}\cdot\vec r} \; \dots \qquad (2.12.)$$

recognize δ-function(s) and assume far-field: $\vec k_{out} = k\frac{\vec r}{r}$, $r \gg r_1, r_2 \Rightarrow |\vec r - \vec r_1| = r + \frac{\vec r\cdot\vec r_1}{r} \Rightarrow G_0 \sim$

$$\Rightarrow \quad G_0(\vec{r},\vec{r}_n) = -\frac{e^{ikr}}{4\pi r}\, e^{-i\vec{k}_{out}\cdot\vec{r}_n}$$

$$\underset{(2.12.)}{\Rightarrow} \quad \psi(\vec{r}) = e^{i\vec{k}_{in}\cdot\vec{r}} - \frac{e^{ikr}}{4\pi r}(2\pi)^3\, t(\vec{k}_{out}, \vec{k}_{in})$$

$$= e^{i\vec{k}_{in}\cdot\vec{r}} + \frac{e^{ikr}}{\cancel{4\pi} r}\, \underbrace{f(\vec{k}_{out}, \vec{k}_{in})}_{\text{scattering amplitude}} \qquad (2.13.)$$

$\rightarrow$ scattering amplitude $f = -2\pi^2\, \tilde{t}$ (2.14.)

probability amplitude for scattering $\vec{k}_{in} \rightarrow \vec{k}_{out}$

2.4 Scattering cross section and optical theorem

cross section $$\boxed{\sigma_{sc}(\vec{k}_{in}) = \int d\Omega_{out}\, |f(\vec{k}_{out}, \vec{k}_{in})|^2}_{|\vec{k}_{out}| = |\vec{k}_{in}|} \qquad (2.15.)$$

In absence of absorption ($\mathrm{Im}\, V \equiv 0$), energy conservation ~~[illegible]~~ implies:

$$\boxed{\sigma_{sc}(\vec{k}_{in}) = \frac{4\pi}{k}\, \mathrm{Im}\, \underbrace{f(\vec{k}_{in}, \vec{k}_{in})}_{\substack{\text{forward scattering}\\ \text{amplitude}}}} \qquad (2.16.)$$

optical theorem

interpretation: energy of scattered wave is taken from the incident wave.

2.5. t-matrix for point sc

point scatterer at $\vec{r}_0$:

Born series (2.10.) $\Rightarrow$

$\underset{\substack{(2.11.)\\(2.15.)}}{\Rightarrow}$ scattering amplitude

with $f = -\frac{1}{4\pi}\int d$

$$\underset{(2.16.)}{\Rightarrow} \quad \sigma_{sc} = 4\pi |f|^2$$

The optical theorem (2.

$$|f|^2 = \frac{1}{k}\, \mathrm{Im}\, f$$

$$\Leftrightarrow \quad f = \frac{1}{k}\, \frac{1}{g(k) + i}$$

2 limiting cases:

(i) resonant scattering

maximum cross secti

$$\Rightarrow \quad \sigma_{sc} \leq \sigma_{max}$$

simplest ansatz: $g(k$

Example: scattering of p

2.5. t-matrix for point scatterers

point scatterer at $\vec{r}_0$: $V(\vec{r}) \neq 0$ only if $|\vec{r}-\vec{r}_0| \ll \lambda$

Born series (2.10.) $\Rightarrow$ $t(\vec{r},\vec{r}\,') \neq 0$ only if $|\vec{r}-\vec{r}_0| \ll \lambda$
$|\vec{r}\,'-\vec{r}_0| \ll \lambda$

$\Rightarrow$ scattering amplitude $f(\vec{k}_{out},\vec{k}_{in}) = e^{i(\vec{k}_{in}-\vec{k}_{out})\cdot\vec{r}_0} \cdot f$ (2.17.)
(2.11.)
(2.15.)

$$\text{with } f = -\frac{1}{4\pi}\int d\vec{r}_1\, d\vec{r}_2\, t(\vec{r}_1,\vec{r}_2)$$

$\Rightarrow$ (2.16.) $\sigma_{sc} = 4\pi |f|^2$ isotropic scattering (s-wave scattering)

The optical theorem (2.17.) implies:

$$|f|^2 = \frac{1}{k}\,\mathrm{Im}\, f$$

$$\Leftrightarrow \quad f = \frac{1}{k}\frac{1}{g(k)+i} \quad , \; g(k) \in \mathbb{R} \text{ arbitrary}$$

2 limiting cases:

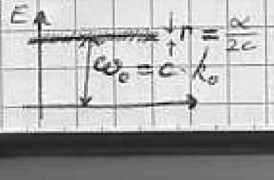
(i) resonant scattering

maximum cross section for $g(k) \equiv 0$

$$\Rightarrow \sigma_{sc} \leq \sigma_{max} = \frac{4\pi}{k^2} = \frac{\lambda^2}{\pi}$$

simplest ansatz: $g(k) = \alpha(k-k_0)$, resonance at $k = k_0$ with width $\frac{1}{\alpha}$

Example: scattering of photon by two-level atom



(ii) Rayleigh-scattering

sphere with radius a and refractive index ε.

point scatterer limit $ak \ll 1$.

Born-approximation (1st term of scattering amplitude) (2.9.)

$$\Rightarrow f = \frac{k^2}{4\pi}\frac{4\pi}{3}a^3(\varepsilon-1) = k^2 a^3 \frac{\varepsilon-1}{3}$$

$$\Rightarrow \sigma_{sc} \propto k^4 \quad \text{(blue sky)}$$

3) Disorder average: diagrammatic technique

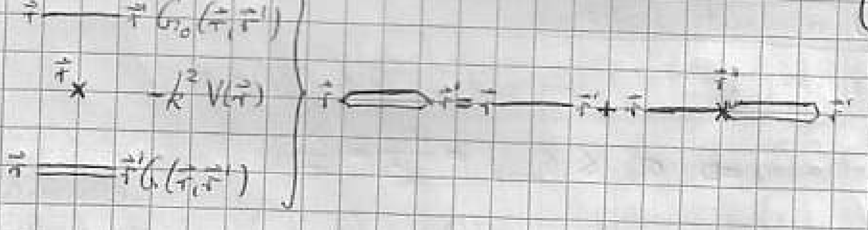
3.1. Average amplitude $\langle\psi(\vec r)\rangle$: Dyson equation

3.1.1. Diagrammatic representation

Lippmann-Schwinger-eq. for $G(\vec r, \vec r')$ (analogous to (2.6.))

$$G(\vec r,\vec r') = G_0(\vec r,\vec r') - k^2\int d\vec r''\, G_0(\vec r,\vec r'')V(\vec r'')\,G(\vec r'',\vec r') \quad (2.18.)$$

(3.1.)



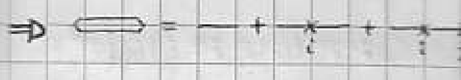
(2.18.)

when connecting two Green's functions

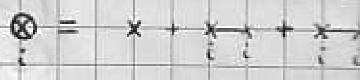
→ integrate over intermediate position.

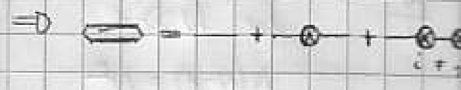
Solution: (Born-series)

$$V = \sum_i V_0(\vec r - \vec r_i)$$

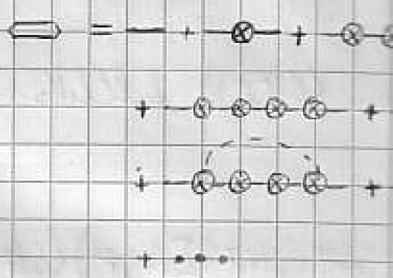


T-matrix of scatterer i:





connect identical scatterers

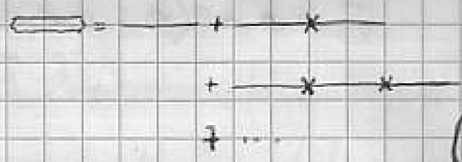


3.1.2. Disorder average

average $\frac{1}{V^N}\int d\vec r_1 \cdots d\vec r_N$

Def.: self-energy $\Sigma(\vec r,$

Solution: (Born-series)



(2.19.)

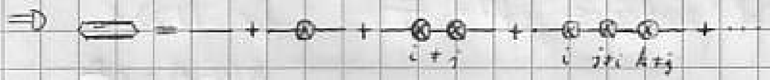
(3.2.)

$$V = \sum_i V_0(\vec{r} - \vec{r}_i)$$

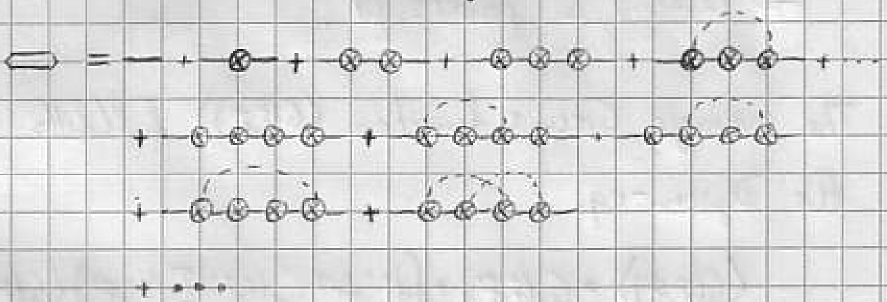
(sum over i, j, ...)

T-matrix of scatterer i:

(Born series)



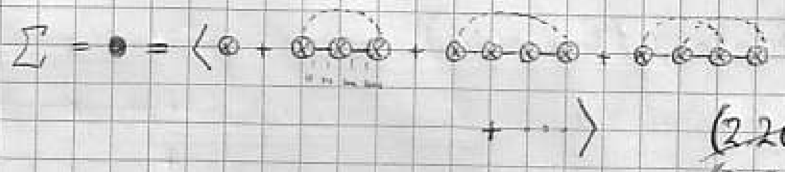
connect identical scatterers by dotted lines



3.1.2. Disorder average

average $\frac{1}{V^N}\int d\vec{r}_1 \cdots d\vec{r}_N$ over positions of all scatterers

Def.: self-energy $\Sigma(\vec{r}, \vec{r}') =$ sum over all "irreducible" diagrams



~~(2.20.)~~
(3.3.)

$$\Leftrightarrow \Sigma(\vec r,\vec r\,') = \frac{N}{V}\int d\vec r_n\, t(\vec r-\vec r_n, \vec r\,'-\vec r_n) + \frac{N(N-1)}{V^2}\iint d\vec r_1 d\vec r_2\, d\vec r\,'' d\vec r\,''' d\vec r\,'''' d\vec r\,'''''$$

$$\times\, t(\vec r-\vec r_1, \vec r\,'-\vec r_1)\, G_0(\vec r\,'', \vec r\,''')\, t(\vec r\,'''-\vec r_2, \vec r\,''''-\vec r_2)\, G_0(\vec r\,'''', \vec r\,''''')$$

$$\times\, t(\vec r\,'''''-\vec r_{12}, \vec r\,'-\vec r_n)$$

$$+ \cdots$$

~~(2.20.)~~
(3.3.)

density of scatterers $\mathcal{N} = \frac{N}{V}$ for $N \gg 1$

$N(N-1) = N^2$

$\Rightarrow$ (2.20.) in powers $\mathcal{N}$.

The average Green's function $\langle G(\vec r,\vec r\,')\rangle$ fullfills the Dyson-eq.

$$\langle G(\vec r,\vec r\,')\rangle = G_0(\vec r,\vec r\,') + \int d\vec r\,'' d\vec r\,''' G_0(\vec r,\vec r\,'')\Sigma(\vec r\,'',\vec r\,''')\langle G(\vec r\,''',\vec r\,')\rangle$$

~~(2.21.)~~
(3.4.)

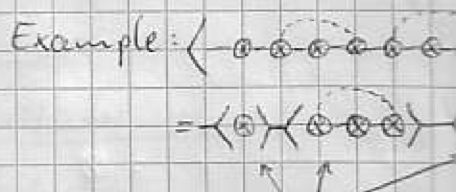
$\Leftrightarrow$ 

"Proof": Iteration 

Every diagram for G

of irreducible diagra

can be averaged s

Example: 

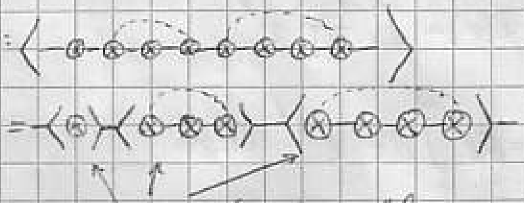
contribute

hence they

"Proof": Iteration 

Every diagram for G factorizes into a product of irreducible diagrams. Unconnected parts can be averaged sepa$\underbrace{\text{rara}}_{\text{ups! iteration ☹}}$tely.

Example:



contribute to self-energy, hence they are contained in 

Tamasito:

3.1.3. Solution of the Dyson equation for an infinite medium

Assume translational invariance (~~for the averaged quantities~~) ⇝ $\Sigma_1(\vec{r}_1,\vec{r}_2) = \Sigma_1(\vec{r}_1-\vec{r}_2)$

FT, see (2.10.):

$$\tilde{\Sigma}(\vec{k}_1,\vec{k}_2) = \frac{1}{(2\pi)^3}\int d\vec{r}_1 d\vec{r}_2\, e^{-i\vec{k}_1\cdot\vec{r}_1 + i\vec{k}_2\cdot\vec{r}_2}\, \Sigma_1(\vec{r}_1-\vec{r}_2)$$

$$= \delta(\vec{k}_1-\vec{k}_2)\,\tilde{\Sigma}(\vec{k}_1) \qquad (3.5.)$$

similarly: vacuum Green's function

$$\tilde{G}_0(\vec{k}_1,\vec{k}_2) = \delta(\vec{k}_1-\vec{k}_2)\,\tilde{G}_0(\vec{k}_1) \qquad (3.6.)$$

where $\tilde{G}_0(\vec{k}_1) = \dfrac{1}{k^2-k_1^2+i\varepsilon}$, $\varepsilon>0,\ \varepsilon\to 0$ (3.7.)

average Green's function

$$\langle \tilde{G}_1(\vec{k}_1,\vec{k}_2)\rangle = \delta(\vec{k}_1-\vec{k}_2)\,\langle \tilde{G}_1(\vec{k}_1)\rangle$$

⇝ Dyson-eq (2.21.):

$$\boxed{\langle \tilde{G}_1(\vec{k}_1)\rangle = \tilde{G}_0(\vec{k}_1) + \tilde{G}_0(\vec{k}_1)\,\tilde{\Sigma}(\vec{k}_1)\,\langle \tilde{G}_1(\vec{k}_1)\rangle}$$

Solution:

$$\langle \tilde{G}_1(\vec{k}_1)\rangle = \frac{\tilde{G}_0(\vec{k}_1)}{1-\tilde{G}_0(\vec{k}_1)\,\tilde{\Sigma}(\vec{k}_1)} = \frac{1}{k^2-k_1^2-\tilde{\Sigma}(\vec{k}_1)} \qquad (3.8.)$$

Fourier-back-transform:

$$\langle G(\vec{r}-\vec{r}')\rangle = -\frac{e^{-ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \qquad (3.9.)$$

same as free propagation but different wave number $\tilde{k}$, defined by

$$\boxed{k^2-\tilde{k}^2-\Sigma(\tilde{k})=0\,,\ \mathrm{Im}\,\tilde{k}>0} \qquad (3.10.)$$

(pole of $\langle\tilde{G}\rangle$)

<u>Def:</u> refractive index

$$\boxed{n=\frac{\tilde{k}}{k}}\in\mathbb{C} \qquad (3.11.)$$

<u>Def:</u> mean free path

$$\boxed{\ell=\frac{1}{2k\,\mathrm{Im}(n)}} \qquad (3.12.)$$

$$\Rightarrow\ |\langle G(\vec{r}-\vec{r}')\rangle|^2=\frac{e^{-\frac{|\vec{r}-\vec{r}'|}{\ell}}}{(4\pi)^2|\vec{r}-\vec{r}'|^2} \qquad (3.13.)$$

Average propagation exponentially damped!

⇝ ℓ: average distance between two scattering events.

Approximate solution for eq.

$$\Rightarrow\ \tilde{k}=\sqrt{k^2-\tilde{\Sigma}(k)}\approx k-$$

$$=k($$

$$\underset{(3.11.)}{\Rightarrow}\quad n=1-\frac{\tilde{\Sigma}(k)}{2k^2}$$

$$\underset{(3.12.)}{\Rightarrow}\quad \ell=-\frac{k}{\mathrm{Im}\,\tilde{\Sigma}(k)}$$

Assumption $|\tilde{\Sigma}|\ll k^2$ implies

<u>3.1.4. Independent scattering app</u>

If $k\ell\gg 1$, Σ is well

term of its series eq(

$$\begin{cases}\bullet=\langle\otimes\rangle\\ \Sigma(\vec{r}_1,\vec{r}_2)=\mathcal{N}\int d\vec{r}_3\end{cases}$$

$$\tilde{\Sigma}(\vec{k}_1,\vec{k}_2)=\mathcal{N}\,\delta($$

$$\underset{\uparrow\atop(2.14.)}{=}\mathcal{N}\,\delta(\vec{k}$$

$$\underset{(3.14.)}{\Rightarrow}\quad n=1-\frac{4\pi\mathcal{N}}{k^2}f(\vec{k},\vec{k})$$

$$\underset{(3.15.)}{\Rightarrow}\quad \ell=\frac{k}{4\pi\mathcal{N}\,\mathrm{Im}\,f(\vec{k},\vec{k})}\underset{\uparrow}{=}$$

optical th

Approximate solution for eq. (3.10.), if $|\tilde{\Sigma}(\vec{k})| \ll k^2$

$$\Rightarrow \tilde{k} = \sqrt{k^2 - \tilde{\Sigma}(k)} \simeq k - \frac{\tilde{\Sigma}(k)}{2k}$$

$$= k\left(1 - \frac{\tilde{\Sigma}(k)}{2k^2}\right)$$

$$(3.11.) \Rightarrow \quad n = 1 - \frac{\tilde{\Sigma}(k)}{2k^2} \qquad (3.14.)$$

$$(3.12.) \Rightarrow \quad l = -\frac{k}{\operatorname{Im} \tilde{\Sigma}(k)} \qquad (3.15.)$$

Assumption $|\tilde{\Sigma}| \ll k^2$ implies $kl \gg 1$ "dilute medium"

3.1.4. Independent scattering approximation

If $kl \gg 1$, Σ is well approximated by the first term of its series eq (3.3.):

$$\begin{cases} \bullet = \langle \otimes \rangle \\ \Sigma(\vec{r}_1, \vec{r}_2) = \mathcal{N} \int d\vec{r}_3 \, t(\vec{r}_1 - \vec{r}_3, \vec{r}_2 - \vec{r}_3) \end{cases}$$

$$\tilde{\Sigma}(\vec{k}_1, \vec{k}_2) = \mathcal{N} \, \delta(\vec{k}_1 - \vec{k}_2) (2\pi)^3 \, \tilde{t}(\vec{k}_1, \vec{k}_2)$$

$$\underset{(2.14.)}{=} \mathcal{N} \, \delta(\vec{k}_1 - \vec{k}_2) \, (-4\pi) \, f(\vec{k}_1, \vec{k}_2)$$

$$\underset{(3.14.)}{\Rightarrow} \quad n = 1 - \frac{4\pi \mathcal{N}}{k^2} f(\vec{k}, \vec{k})$$

$$\underset{(3.15.)}{\Rightarrow} \quad l = \frac{k}{4\pi \mathcal{N} \operatorname{Im} f(\vec{k}, \vec{k})} \underset{\text{optical theorem (2.16.)}}{=} \frac{1}{\mathcal{N} \sigma} \qquad (3.16.)$$

3.2. Average ($\langle|\psi|^2\rangle$) intensity

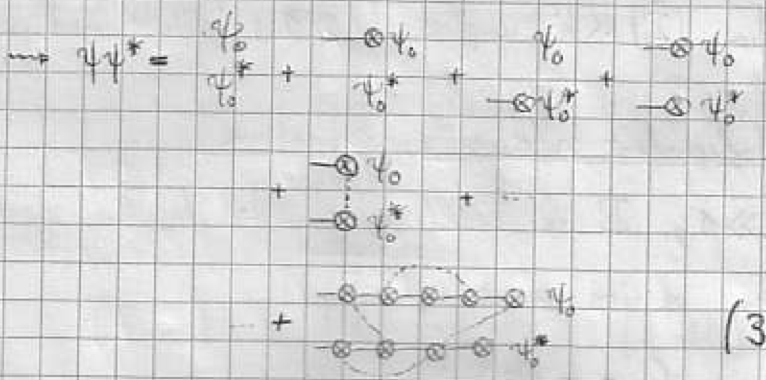
3.2.1. Intensity diagrams

Born series (2.7.) for $\psi(\vec{r})$

$$\psi = \psi_0 + \underset{i}{-\otimes}\,\psi_0 + \underset{i}{-\otimes}\underset{i\neq j}{-\otimes}\,\psi_0 + \dots \qquad (3.17.)$$

Complex conjugate → Born series for ψ^*

We write the series for $\psi\psi^*$ as ~~###~~ two lines



(3.18.)

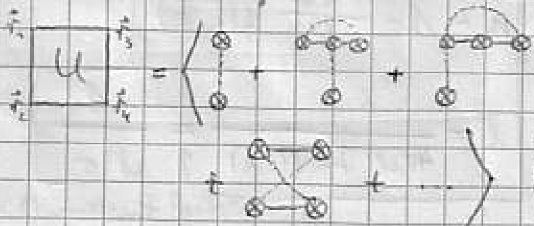
3.2.2. Bethe-Salpeter equation

Disorder average of (3.18.)

Def: Intensity operator

$U(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4)$ = sum of all irreducible diagrams

Symbol



(3.19.)

Bethe-Salpeter-eq.:

$$\langle\psi\psi^*\rangle = \langle\psi\rangle\langle\psi^*\rangle +$$

$$\Leftrightarrow \langle\psi(\vec{r}_1)\psi^*(\vec{r}_2)\rangle = \langle\psi(\vec{r}_1)\rangle\langle\psi^*(\vec{r}_2)\rangle + \int d$$

x

"Proof" (Motivation): Iteration

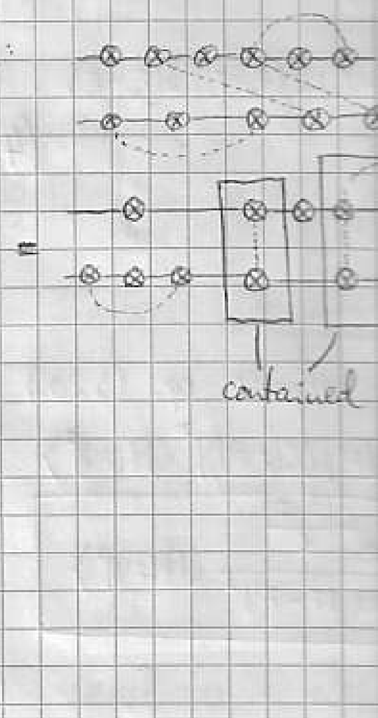
$$\langle\psi\psi^*\rangle = \langle\psi\rangle\langle\psi^*\rangle +$$

Every intensity diagram

a product of irreducible

Green's functions.

e.g.:



Bethe-Salpeter-eq.:

$$\langle\psi\psi^*\rangle = \langle\psi\rangle\langle\psi^*\rangle + \boxed{}\boxed{U}\,\langle\psi\psi^*\rangle \qquad (3.20.)$$

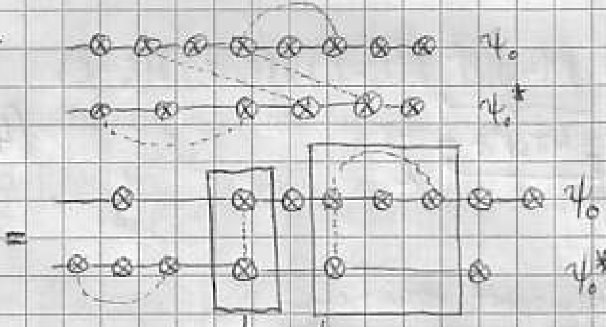
$$\Rightarrow \langle\psi(\vec r_1)\psi^*(\vec r_2)\rangle = \langle\psi(\vec r_1)\rangle\langle\psi^*(\vec r_2)\rangle + \int d\vec r_3\, d\vec r_4 \cdots d\vec r_6 \,\langle G(\vec r_1,\vec r_3)\rangle\langle G(\vec r_2,\vec r_4)\rangle\, U(\vec r_3,\vec r_4,\vec r_5,\vec r_6)$$
$$\times\langle\psi(\vec r_5)\psi^*(\vec r_6)\rangle$$

"Proof" (Motivation): Iteration

$$\langle\psi\psi^*\rangle = \langle\psi\rangle\langle\psi^*\rangle + \boxed{}\boxed{U}\,\langle\psi\rangle\langle\psi^*\rangle + \boxed{}\boxed{U}\boxed{}\boxed{U}\begin{matrix}\langle\psi\rangle\\ \langle\psi^*\rangle\end{matrix} + \cdots$$

Every intensity diagram factorizes into a product of irreducible diagrams and average Green's functions.

e.g.:



contained in U, see (3.19.)

4) Incoherent transport

4.1. Independent scattering approximation

For $kl \gg 1$ (dilute medium), we may approximate: $\Sigma = \langle \otimes \rangle$ (see (3.14))

$$U = \left\langle \begin{array}{c} \otimes \\ \vdots \\ \otimes \end{array} \right\rangle$$

Intensity operator for point scatterers

$$U(\vec r_1, \vec r_2, \vec r_3, \vec r_4) = \mathcal{N} \int d\vec r \, \underbrace{t(\vec r_1 - \vec r, \vec r_3 - \vec r)}_{\overset{(2.11),(2.14),(2.17)}{=} \delta(\vec r - \vec r_1)\,\delta(\vec r_3 - \vec r)(4\pi f)} t^*(\vec r_2 - \vec r, \vec r_4 - \vec r)$$

$$= \underbrace{\mathcal{N}(4\pi)^2 |f|^2}_{= 4\pi \mathcal{N} \sigma \overset{(3.16)}{=} \frac{4\pi}{l}} \delta(\vec r_2 - \vec r_1)\,\delta(\vec r_3 - \vec r_1)\,\delta(\vec r_4 - \vec r_1) \qquad (4.1.)$$

Incoherent transport equation

Insert (4.1.) into Bethe - Salpeter - eq. (3.20.)

$$\langle |\psi(\vec r)|^2 \rangle = |\langle \psi(\vec r) \rangle|^2 + \frac{4\pi}{l} \int d\vec r' \, |\langle G(\vec r, \vec r') \rangle|^2 \langle |\psi(\vec r')|^2 \rangle$$

$$\boxed{\langle |\psi(\vec r)|^2 \rangle \overset{(3.13.)}{=} |\langle \psi(\vec r) \rangle|^2 + \int d\vec r' \, \frac{e^{-\frac{|\vec r - \vec r'|}{l}}}{4\pi l \, |\vec r - \vec r'|^2} \langle |\psi(\vec r')|^2 \rangle} \qquad (4.2.)$$

4.2. Diffusion approximation

Assume Intensity $J(\vec r) = \langle |\psi$

lengthscale l.

$$\Rightarrow J(\vec r + \vec r') = \underbrace{J(\vec r)}_{(0)} + \underbrace{\vec r' \cdot \nabla J(\vec r)}_{(1)}$$

Insert into transport eq.

$$J(\vec r) = J_{coh}(\vec r) + \underbrace{\int d\vec r' \, \frac{e^{-r'/l}}{4\pi l \, r'^2}}_{\text{Shifte}}$$

$$(0): \quad J(\vec r) \int d\vec r' \, \frac{e^{-r'/l}}{4\pi l \, r'^2} = J(\vec r)$$

(1): vanishes due to sy

$$(2): \quad \frac{1}{2} \int d\vec r' \, \frac{e^{-r'/l}}{4\pi l \, r'^2} (x'^2 \partial_x^2 + y'^2$$

$$= \frac{1}{6} \Delta J(\vec r) \underbrace{\int d\vec r' \, \frac{e^{-r'/l}}{4\pi l \, r'^2}}_{2l^2}$$

$\Rightarrow$ In total: $\boxed{\frac{l^2}{3} \Delta J(\vec r) +}$

stationary diffusion

source term $s(\vec r) =$

4.2 Diffusion approximation

Assume Intensity $J(\vec r) = \langle |\psi(\vec r)|^2 \rangle$ varies slowly on lengthscale ℓ.

$$\Rightarrow J(\vec r + \vec r\,') = \underbrace{J(\vec r)}_{(0)} + \underbrace{\vec r\,' \cdot \nabla J(\vec r)}_{(1)} + \underbrace{\tfrac{1}{2}(\vec r\,' \cdot \nabla)^2 J(\vec r)}_{(2)} \qquad (4.3.)$$

Insert into transport eq.

$$J(\vec r) = J_{coh}(\vec r) + \underbrace{\int d\vec r\,' \frac{e^{-r'/\ell}}{4\pi \ell r'^2} J(\vec r + \vec r\,')}_{\text{shifted variable of integration}} \qquad (4.4.)$$

$$(0): \quad J(\vec r) \int d\vec r\,' \frac{e^{-r'/\ell}}{4\pi \ell r'^2} = J(\vec r)$$

(1): vanishes due to symmetry

$$(2): \quad \frac{1}{2} \int d\vec r\,' \frac{e^{-r'/\ell}}{4\pi \ell r'^2} \left(x'^2 \partial_x^2 + y'^2 \partial_y^2 + z'^2 \partial_z^2 \right) J(\vec r)$$

$$= \frac{1}{6} \Delta J(\vec r) \underbrace{\int d\vec r\,' \frac{e^{-r'/\ell}}{4\pi \ell r'^2} r'^2}_{2\ell^2} = \frac{\ell^3}{3} \Delta J(\vec r)$$

$$\Rightarrow \text{In total:} \quad \boxed{\frac{\ell^2}{3} \Delta J(\vec r) + J_{coh}(\vec r) = 0} \qquad (4.5.)$$

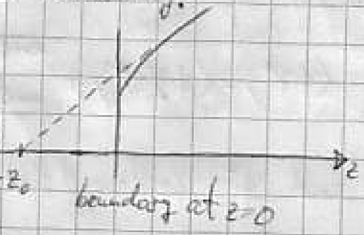
stationary diffusion equation, with source term $s(\vec r) = J_{coh}(\vec r)$

Tomasito:

Boundary conditions:

Solution of transport equation (4.2.) exhibits the following behaviour at the boundary:

$$J(z)\big|_{z=0} - z_0 \partial_z J(z)\big|_{z=0} = 0 \quad (4.6.)$$



Extrapolation length $z_0 = 0.71\, l$ (in 3D)

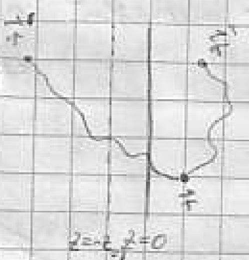
Diffusion propagator: $\frac{l^2}{3} \Delta P(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}')$

Solution for an infinite medium:

$$P_\infty(\vec{r} - \vec{r}') = \frac{3}{l^2 4\pi |\vec{r} - \vec{r}'|}$$

Diffusion with boundary:

$$\boxed{P(\vec{r}, \vec{r}') = P_\infty(\vec{r}, \vec{r}') - P_\infty(\vec{r}, \vec{r}'')} \quad (4.7.)$$



fulfills diffusion eq. (4.5.)

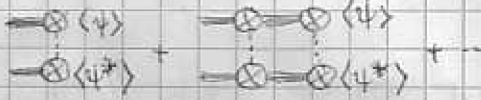
and boundary conditions.

Ladder diagrams:

In independent scatterer approximation, transport of average intensity is described by ladder diagrams

Iteration of Bethe-Salpeter-eq gives:

$$\langle\psi\psi^*\rangle = \langle\psi\rangle\langle\psi^*\rangle + \dots$$



⇒ ψ and ψ* follow the same scattering path.

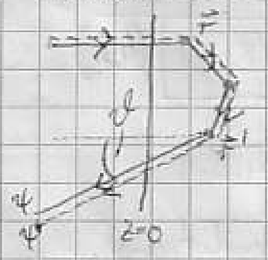
→ neglect interferences between different scattering paths

5) Coherent ~~back scattering~~ transport

5.1. Coherent back scattering

back-scattered intensity $\gamma(\vartheta)$

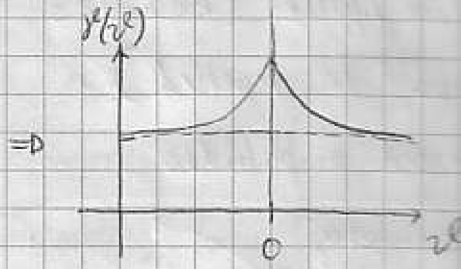
- incoherent contribution: ladder diagrams



$$\gamma_L(\vartheta) = \int d\vec r\, d\vec r'\, \underbrace{e^{-z/l}}_{\text{incoming wave}}\ \underbrace{P(\vec r,\vec r')}_{\text{diffusion from } \vec r \text{ to } \vec r'}\ \underbrace{e^{-z'/l\cos(\vartheta)}}_{\text{outgoing wave}}$$

- coherent contribution

⇒



$$\gamma_C(\vartheta) = \int d\vec r\, d\vec r'\, e^{-z\left(\frac{1}{2l} +\right.}$$

$$\times P(\vec r,\vec r')\ e$$

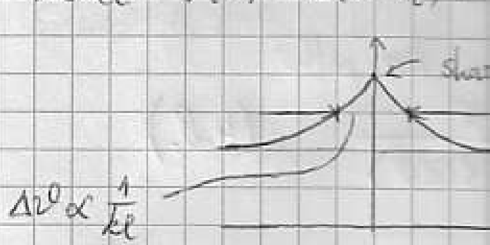
, with $\vec k_{in} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$,

Using diffusion approximat

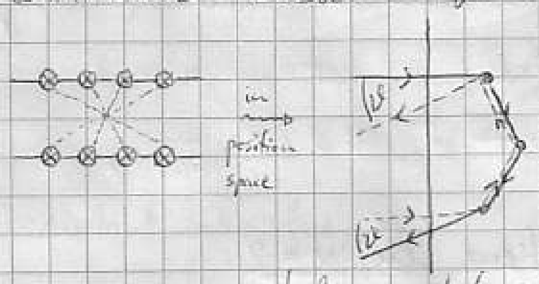
we can calculate (5.2

(→ Akkermans, PRL (198

Result: $\gamma(\vartheta) = 2(l+z_0) +$

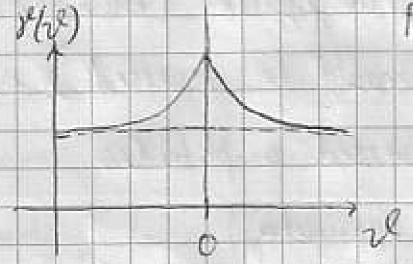


- coherent contribution: crossed diagrams



⇒ interference between reversed paths; for $\vartheta = 0$: constructive interference

⇒



$$\gamma_c(\vartheta) = \int d\vec{r}\, d\vec{r}'\, e^{-z\left(\frac{1}{2l} + \frac{1}{2l\cos\vartheta}\right)}\, e^{i(\vec{k}_{in} + \vec{k}_{out})(\vec{r} - \vec{r}')}$$
$$\times P(\vec{r}, \vec{r}')\, e^{-z'\left(\frac{1}{2l} + \frac{1}{2l\cos(\vartheta)}\right)} \qquad (5.2.)$$

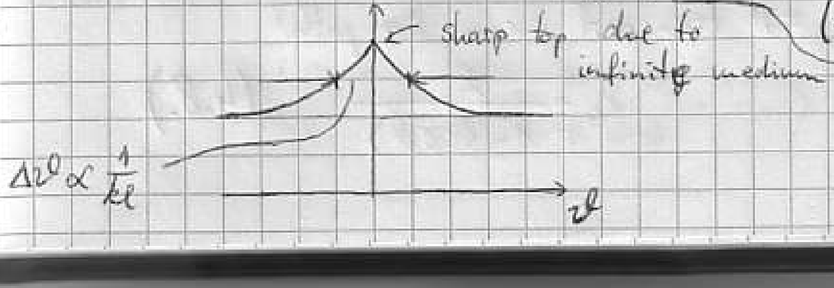
, with $\vec{k}_{in} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$, $\vec{k}_{out} = \begin{pmatrix} k \sin\vartheta \\ 0 \\ -k\cos\vartheta \end{pmatrix}$

Using diffusion approximation for $P(\vec{r}, \vec{r}')$, eq. (4.7.), we can calculate (5.2) analytically

(→ Akkermans, PRL (1986))

Result: $\gamma(\vartheta) = 2(l + z_0) + \frac{1 - e^{-2k(l+z_0)|\vartheta|}}{k|\vartheta|}$, for $\vartheta \ll 1$

(5.3.)



- CBS enhancement factor: $\alpha = \left.\frac{\gamma_L + \gamma_C}{\gamma_L}\right|_{\vartheta=0} \leq 2$

$\uparrow$ two wave-interference

conditions for $\alpha = 2$:

- single scattering negligible (enhances background, can't be reversed)
- equality of reversed amplitudes (reciprocity symmetry)

Breaking of reciprocity symmetry possible due to:

- for light scattering: incoming and outgoing polarization not identical
- moving scatterers

5.2. weak localization

5.2.1. Anisotropic random walk

scattering amplitude f depends on ϑ.

$\Rightarrow$ transport mean free path

$$l_{tr} = \frac{l}{1 - \langle \cos\vartheta \rangle} \qquad (4.8.)$$

describes the length sca

about incident direction

$\rightsquigarrow$ Diffusion constant $D=$

5.2.2. Reduction of the

~~Enhancement of back sca~~

According to eq. (4.8.) e

scattering leads to reduc

mean free path and

reduction of the diff

$$\frac{\delta D}{D} \propto - \frac{1}{(kl)^2}$$

5.2.3. Diagrammatic descrip

The intensity ~~propagator~~ operat

by $U = \left\langle \overset{\otimes}{\underset{\otimes}{\vdots}} \right\rangle$.

Extend this by crossed

$U = \left\langle \overset{\otimes}{\underset{\otimes}{\vdots}} + \right.$

describes the length scale on which the information about incident direction is lost.

$\leadsto$ Diffusion constant $D = \frac{1}{3} l_{tr} v_{tr}$ (4.9.)

transport velocity

5.2.2. Reduction of the mean free path

~~Enhancement of back scattering~~

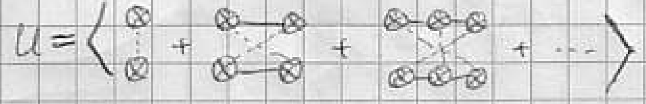
According to eq. (4.8.) enhancement of back scattering leads to reduction of transport mean free path and through eq. (4.9.) to a reduction of the diffusion constant:

$$\frac{\delta D}{D} \propto - \frac{1}{(kl)^2} \quad \text{(in 3D)} \qquad (5.4.)$$

5.2.3. Diagrammatic description

The intensity ~~propagator~~ operator was approximated by $U = \left\langle \begin{smallmatrix}\otimes \\ \vdots \\ \otimes\end{smallmatrix} \right\rangle$.

Extend this by crossed diagrams:



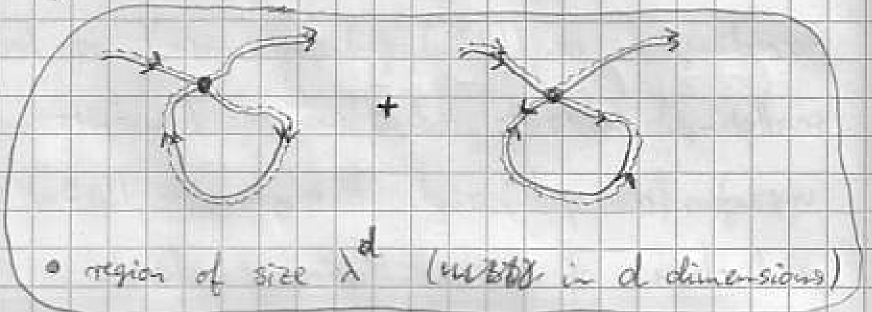
Insert U into Bethe-Salpeter-eq. (3.20.)

Apply diffusion approximation

$$\longrightarrow l_{tr} \approx l\left(1-\frac{3}{2(kl)^2}\right)$$

5.2.4. Return probability

Contribution of weak localization is proportional to the "return probability" of the diffusive path, $P(\vec{r},\vec{r}')$, where $|\vec{r}-\vec{r}'| < \lambda$.



• region of size λ^d (in d dimensions)

⇝ constructive interference between reversed paths increases return probability → diffusion decreases.

$$\frac{\delta D}{D} \approx -\frac{1}{(kl)^2} \quad \text{(in 3D)}$$

in 1D and 2D: $\delta D \to -\infty$, i.e. diverges

(⇝ strong localization)

5.3. Anderson or strong

5.3.1. Anderson model

$$H = \sum_{ij} \underset{\substack{\uparrow \\ \text{transitions}}}{T_{ij}} |i\rangle\langle j| + \sum_i$$

onsit

$|i\rangle$: electron a

T_{ij} transitions from i

$T_{ij} = \{$

Discovery of P.W. Anders

Eigenfunctions of H

i.e. $|\langle i|\psi\rangle|^2 \propto e$

⇒ no diffusive transport

interferences.

For 3D lattice: Andersen

meta

extended eigenfu

5.3. Anderson or strong localization

5.3.1. Anderson model

$$H = \sum_{ij} T_{ij} \, |i\rangle\langle j| + \sum_{i} \varepsilon_i \, |i\rangle\langle i| \qquad (5.4.)$$

$\uparrow$ transitions $\qquad \uparrow$ onsite energy (random in $[-\frac{W}{2}, \frac{W}{2}]$)

$|i\rangle$: electron at site i (eg. 1D-lattice:

)

T_{ij} transitions from i to j:

$$T_{ij} = \begin{cases} T & \text{, if } i,j \text{ are neighbors} \\ 0 & \text{, else} \end{cases}$$

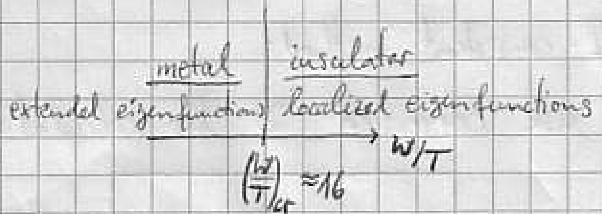
Discovery of P.W. Anderson (1958):

Eigenfunctions of H may be exponentially localized,

i.e. $|\langle i|\Psi\rangle|^2 \propto e^{-r_i/\xi}$, r_i distance of i to center of $|\Psi\rangle$.

$\Rightarrow$ no diffusive transport due to destructive interferences.

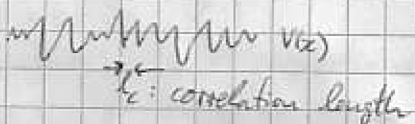
For 3D lattice: Anderson transition



For 1D and 2D lattice eigenfunctions are always localized.

Anderson localization also occurs in continuous systems:

e.g. 1D wave: $\psi''(x) + k^2(1+V(x))\psi(x) = 0$



$\rightsquigarrow$ localization for $kl_c < 1$.

3D: localization expected if

$$k\underset{\uparrow}{l} \lesssim 1 \quad \text{(Ioffe-Regel-criterion)} \quad (5.5.)$$

transport mean free path

5.3.2. Self-consistent theory of localization

Idea: weak localization is precursor of strong one.

<u>Reminder</u>: $\delta D = D_0 - D \propto P_{D_0}(\vec{r},\vec{r})$ (weak)

return probability for classical (Boltzmann) diffusion constant.

Self-consistent method:

use corrected D for calculation of P

$$\Rightarrow \qquad \boxed{D_0 - D \propto P_D(\vec{r},\vec{r})}$$

Results of this method ag

"scaling theory" (differ

e.g. metal-insulator-

---

self-consistent and scal

of describing Anderson

Results of this method agree with the results of "scaling theory" (different approach).

e.g. metal-insulator-transition in 3D

---

self-consistent and scaling theory are incapable of describing Anderson localization completely.