Entanglement and random quantum states

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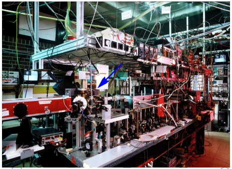
Outline

Quantum information

Quantum feats :

- Quantum secure communication
  (no entanglement required, just no cloning)
- Teleportation
  (entanglement needed, e.g., EPR state)
- Quantum computation
  (sufficient entanglement necessary (but not sufficient), else efficient classical simulation possible)



Hilbert space

Hilbert space

$$\mathcal{H} = \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$$

Usually we talk about qubits as basic units:

- system with two levels $|0\rangle$ and $|1\rangle$; 2 dimensional Hilbert space :
  - spin $\frac{1}{2}$ particle (electron) : two orthogonal states are spin up and spin down
  - photon polarization : two linear (circular) polarizations
  - two energy states of an ion
- Whole system of n qubits : Hilbert space is $\mathcal{H} = \mathcal{H}_i^{\otimes n}$, $\dim(\mathcal{H}) = 2^n$ (exponential in n)
- Elements from Hilbert space in computational basis $|01\ldots1\rangle = |0\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle$.

Definition of a separable state

- Definition of a separable state:

Pure states

$$|\psi\rangle = |\psi^{\mathrm{A}}\rangle \otimes |\psi^{\mathrm{B}}\rangle$$

Mixed states (density matrices)

$$\rho = \sum_i p_i |\psi_i^{\mathrm{A}}\rangle\langle\psi_i^{\mathrm{A}}| \otimes |\psi_i^{\mathrm{B}}\rangle\langle\psi_i^{\mathrm{B}}|$$

$p_i > 0$ and $\sum_i p_i = 1$ ($|\psi_i^{\mathrm{A,B}}\rangle$ need not be orthogonal)

Entangled states

A state is entangled if it is not separable.

Basis states $|0\rangle$ and $|1\rangle$ (aka. quantum bits - qubits).

- Pure entangled state of two qubits:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad |\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Bell or EPR states.

Random quantum states - motivation

Analogy:
(classical) random numbers $\Longleftrightarrow$ (quantum) random states

Why study?

- They are generic (typical state).
- Complex quantum system - random state during evolution (quantum chaos).
- Shared entangled state is a useful resource!
(state with a large Schmidt rank, e.g., random, maximally entangled...)

Random quantum states (def.)

Random pure states - definition

Several possibilities to define random $|\psi\rangle = \sum_i c_i |i\rangle$:

- c_i are random Gaussian complex numbers
- $|\psi\rangle$ is eigenvector of a random Hermitian matrix
- $|\psi\rangle$ is a column of a random unitary matrix
  - unique unitarily invariant Haar measure

Questions

1. What are their entanglement properties?
2. How to generate them?

Entanglement of pure states

Pure state entanglement

Schmidt decomposition:

$$|\psi\rangle = \sum_{i=0}^{N_A-1} \sqrt{\lambda_i}|w_i^A\rangle \otimes |w_i^B\rangle.$$

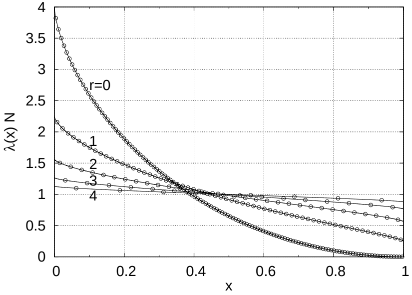
- $|w_i^A\rangle$ and $|w_i^B\rangle$ are orthonormal
- λ_i are eigenvalues of the reduced $\rho_A = \mathrm{tr}_B|\psi\rangle\langle\psi|$

- For mixed states it is hard to quantify entanglement
- For pure states easy : all λ_i completely characterize it
  - if all equal, $\lambda_i = \frac{1}{N_A}$, "the most" entangled state; in 2×2 this is for instance EPR state

Can we calculate λ_i for random pure states?

Eigenvalues for random states

To calculate average $\langle \lambda_i \rangle$ (average over random states) in the limit $N_\mathrm{A} \to \infty$ use Marčenko-Pastur for the density of eigenvalues (Žnidarič, JPA 40 F105 '07)



- $w = 1/2^{2r} = N_\mathrm{A}/N_\mathrm{B}$ (bipartition to $n/2 - r$ and $n/2 + r$ spins)
- $w \ll 1 \Longrightarrow \rho_\mathrm{A} \approx \frac{1}{N_\mathrm{A}} \mathbb{1}$
- Deviations from $\lambda_i = 1/N_\mathrm{A}$ are $\sim \frac{2}{N_\mathrm{A}}\sqrt{w}$, *i.e.*, exponentially small in the number of "particles" in $\mathcal{H}_\mathrm{B}$.

How to generate random states?

- In principle we need $2N - 1$ parameters for random $|\psi\rangle$ (too many) They are generic, but are they physical?
- We want a method that is polynomial in $n = \log(N)$

Example

- start with a non-random $|\psi\rangle$, *e.g.*, $|00\ldots0\rangle$
- at each step apply a random 2-qubit gate to a random pair of qubits

How many steps do we need?

Number of steps

Number of steps until all eigenvalues $\approx 1/N_{\mathrm{A}}$, purity $I = \mathrm{tr}_A \rho_{\mathrm{A}}^2 \approx 1/N_{\mathrm{A}}$? ($|\psi\rangle$ is as entangled as a typical random state)

Single step analysis

- expand $\rho = |\psi\rangle\langle\psi|$ over Pauli basis, $\rho(c_i) = \sum_i c_i\ \sigma^{i_1} \otimes \sigma^{i_2} \otimes \cdots \otimes \sigma^{i_n}$
- $\sigma^{i_j} \in \{\mathbb{1}, \sigma^x, \sigma^y, \sigma^z\}$, matrix basis for $U(2)$.
- after one step you get $\rho'(c_i') = U\rho(c_i)U^\dagger$
- to calculate purity we need c_i^2
- it turns out that $(c_i')^2$ depend linearly on $(c_i)^2$ (no $c_i c_j$ terms)!
- Markov chain, $(c')^2 = M \cdot c^2$ (Oliveira, Dahlstein, Plenio, PRL 98, 130502 (07))

Markov chain

Markov chain

- Markov chain only if two-qubit gate preserves Pauli matrices ($W\sigma^{\alpha}W^{\dagger}=\sigma^{\beta}$)
- dimension of M is 4^n
- What is the gap Δ? $\longrightarrow$ number of needed steps
- Is the chain rapidly mixing, *i.e.*, $\Delta \sim 1/\mathrm{poly}(n)$?

- Analytical estimate for $W=\mathrm{CNOT}$ and random $i-j$ coupling: $\Delta > \frac{4}{9n(n-1)}$ (Oliveira *et.al.* (07))
- Numerics gives (Žnidarič, PRA **76**, 012318 (07)) $\Delta \asymp 1.6/n$.

Analytical solution

Space of n "qudits", e.g., each site 4 states (Pauli matrices).

$$M = \frac{1}{n} \sum_{i}^{n} T_{i,i+1} \otimes \mathbb{1}$$

T transition matrix for two "qudits" ($4^2 \times 4^2$) and $U(4)$ gate,

$$T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{15} & \cdots & \frac{1}{15} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{15} & \cdots & \frac{1}{15} \end{pmatrix}.$$

Calculate the gap Δ!

Analytical solution (cont.)

Markov chain on 4^n equivalent to spin chain on 2^n

$U(4)$ and nearest neighbor coupling – XY model:

$$h_{\mathrm{XY}} = \frac{1+\gamma}{2}\sigma_i^{\mathrm{x}}\sigma_j^{\mathrm{x}} + \frac{1-\gamma}{2}\sigma_i^{\mathrm{y}}\sigma_j^{\mathrm{y}} + h(\frac{1}{2}\sigma_i^{\mathrm{z}} + \frac{1}{2}\sigma_j^{\mathrm{z}}).$$

$U(4)$ and all-all coupling – Lipkin-Meshkov-Glick:

$$hS_{\mathrm{z}} + J_{\mathrm{x}}S_{\mathrm{x}}^2 + J_{\mathrm{y}}S_{\mathrm{y}}^2$$

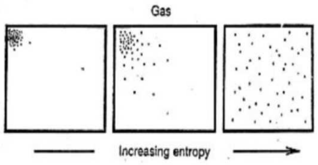
CNOT and XY gates – XYZ model

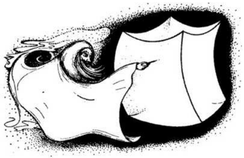
Analytical gap $\Delta \sim \frac{1}{n}$

Entanglement and classicality

Question

1. Why is there no observable entanglement in macro-world?





Classical irreversibility:

- practical issues of reversibility : almost impossible to reverse
- role of initial conditions: for most entropy increases

picture from R.Penrose

Practicality

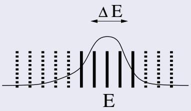
Random states are quantum

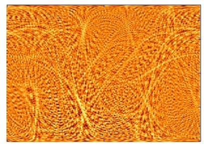
- almost maximally entangled, von Neumann entropy $S \approx \frac{n}{2}$

  random states are very entangled - very quantum

...are classical

- in classical limit ($N \to \infty$) random states mimic microcanonical density
- quantum expectation value in a random state is close to the classical average





Paradox

How come?

Resolution:

- von Neumann entropy does not tell everything!
- Entanglement hidden in many degrees of freedom, e.g., Schmidt coefficients are $\sim 1/\sqrt{N_A}$ - exponentially small.
- Difficult to detect!

For all practical purposes classical.

Entanglement Witness

Definition

- If $\mathrm{tr}(\rho_{\mathrm{sep}} W) > 0$ for all separable ρ_{sep} and $\mathrm{tr}(\rho_{\mathrm{ent}} W) < 0$ for at least one entangled ρ_{ent} W is an entanglement witness. It detects entanglement of ρ_{ent}.
- In general different W for different ρ_{ent}.

Decomposable EW

Especially simple are decomposable EW:

$$W = P + Q^{\mathrm{T_B}}, \qquad P, Q \geq 0$$

- $Q^{\mathrm{T_B}}$ is partial transposition with respect to subspace B
- D-EW are equivalent to PPT criterion

Example of W

Example

- Take for Q a projector, $Q = |GHZ\rangle\langle GHZ|$ with $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and $P = 0$.
- Subsystem B is last qubit, $W = Q^{T_B}$, $W = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111| + |001\rangle\langle 110| + |110\rangle\langle 001|)$.
- W has one negative eigenvalue with the eigenvector $|\psi\rangle = \frac{1}{2}(|001\rangle - |110\rangle)$.
- $\langle\psi|W|\psi\rangle = -\frac{1}{2}$. Detects entanglement of $|\psi\rangle$.
- $\langle GHZ|W|GHZ\rangle = \frac{1}{2}$. Does not detect entanglement of $|GHZ\rangle$.

Results (M.Ž., T.Prosen, G.Benenti and G.Casati, JPA **40**, 13787 (2007))

- Large random states almost classical.
- **Random** W (unknown ρ) : Gaussian $p(w)$, $\mathrm{tr}(W\rho) \sim -1/N_A^2$
  - $\mathcal{P}(w<0) = (1-\mathrm{erf}(1/\sqrt{2}))/2 \approx 0.16$
  - mixing k states, $\rho \sim \sum^k |\psi_i\rangle\langle\psi_i|$,
    $\mathcal{P}(w<0) = (1-\mathrm{erf}(\sqrt{k/2}))/2 \asymp \frac{1}{\sqrt{k}}\mathrm{e}^{-k/2}$
- **Optimal** W (known ρ) : $\mathrm{tr}(W\rho) = -|\lambda_{\mathrm{min}}(\rho^{\mathrm{T_B}})|$
  - pure state $(k=1)$: $\lambda_{\mathrm{min}} = -4/N_A$
  - large $k \gg 1$: $\lambda_{\mathrm{min}} \sim -1/N_A^2$
  - $k > k^* \approx 4N_A^2$: $\lambda_{\mathrm{min}} > 0$

Initial conditions

Setting

- Large n qubit quantum system
- Start in generic separable state (no entanglement)
- Evolve with some hamiltonian
- What is entanglement of smaller subsystem (two qubits)

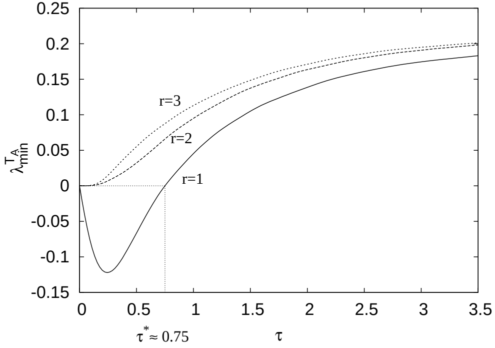
How much entanglement, for how long...?

We would "like" to see: For generic i.c. low entanglement only for short times and regardless of H!

(M.Ž. preprint arXiv:0805.0523)

Arbitrary H with two-particle coupling h. Initial time scale dictated by

$$\lambda_{\mathrm{min}}^{T_A} = -|\delta| t + \mathcal{O}(t^2), \qquad \delta = \langle \chi_A^{\perp} \chi_B^{\perp} | h^{(2)} | \chi_A \chi_B \rangle.$$



- n.n. two-body RMT model
- distance between qubits r
- universality : almost the same dependence for any H

Initial state randomness as a universal source of decoherence

- randomness in initial state
- leads to universal behavior of entanglement between two qubits regardless of the coupling
- entanglement present only for short time and directly coupled qubits

Summary

- Giving Schmidt coefficients completely determines entanglement of pure states – analytical expression
- Generating random bipartite entanglement in $\tau \sim n \ln \frac{1}{\epsilon}$, gap $\Delta \sim 1/n$

No entanglement in systems with many degrees of freedom:

- Practicality : hard to detect because many small Schmidt coefficients
- Generic initial states : entanglement only for short times and directly coupled qubits. Independent of H!