Microwave studies of chaotic systems

Lecture 3: A random matrix approach to fidelity

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Contents

- Global perturbations
- Local perturbations
- Supersymmetries for pedestrians

Global perturbations

Definition of fidelity

Introduced by Peres 1985 as a measure of the stability of quantum motion:

Perturbed Hamiltonian: $H_\lambda = H_0 + \lambda V$

Time evolution of same initial state $|\psi_0>$ under H_0 and H_λ:

$$|\psi_0(t)> = e^{-2\pi\imath H_0 t}|\psi_0> \qquad |\psi_\lambda(t)> = e^{-2\pi\imath H_\lambda t}|\psi_0>$$

The fidelity amplitude is defined as the overlap integral

$$f_\lambda(t) = <\psi_0(t)\,|\,\psi_\lambda(t)>$$

and the fidelity is $F_\lambda(t) = |f_\lambda(t)|^2$.

The three regimes

- $\lambda \ll 1$: perturbative regime

  Gaussian decay

- $\lambda \approx 1$: Fermi golden-rule regime

  Exponential decay depending on the perturbation strength

  - semiclassical approach (Cerrutti, Tomsovic 2002)
  - random matrix approach (Gorin, Prosen, Seligman 2002)

- $\lambda \gg 1$: Lyapunov regime

  Exponential decay independent on the perturbation strength

  - First observed in NMR spin-echo experiments (Zhang *et al.* 1992)
  - explanation by Jalabert, Pastawski 2001

Linear-response results

Linear-response result by Gorin, Prosen, Seligman 2004

$$f_\lambda(t) = 1 - 4\pi^2\lambda^2\, C(t)$$

where

$$C(t) = t^2 + \tfrac{t}{2} - \int_0^t \int_0^\tau b_2(t')dt'd\tau\,,$$

$b_2(t)$: two-point form factor.

Exponentiated linear-response result, to extend validity:

$$f_\lambda(t) \sim e^{-4\pi^2\lambda^2\, C(t)}\,.$$

Describes fidelity decay both in the perturbative and the Fermi golden rule regime.

Beyond linear response

Using supersymmetry techniques exact results are obtainable (Stöckmann, Schäfer 2004), yielding for the GUE

$$f_\epsilon(\tau) = \begin{cases} e^{-\frac{1}{2}\epsilon\tau}\left[s(\frac{1}{2}\epsilon\tau^2) - \tau s'(\frac{1}{2}\epsilon\tau^2)\right], & \tau \le 1 \\ e^{-\frac{1}{2}\epsilon\tau^2}\left[s(\frac{1}{2}\epsilon\tau) - \frac{1}{\tau}s'(\frac{1}{2}\epsilon\tau)\right], & \tau > 1 \end{cases},$$

where

$$s(x) = \frac{\sinh(x)}{x}.$$

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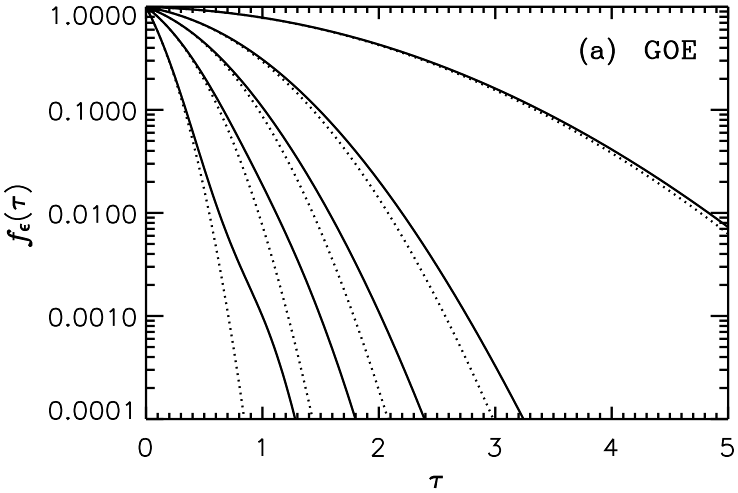
where

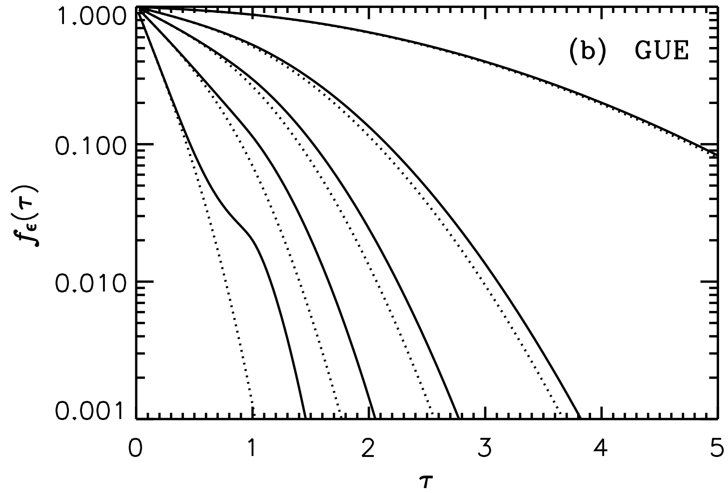
$$s(x) = \frac{\sinh(x)}{x}.$$

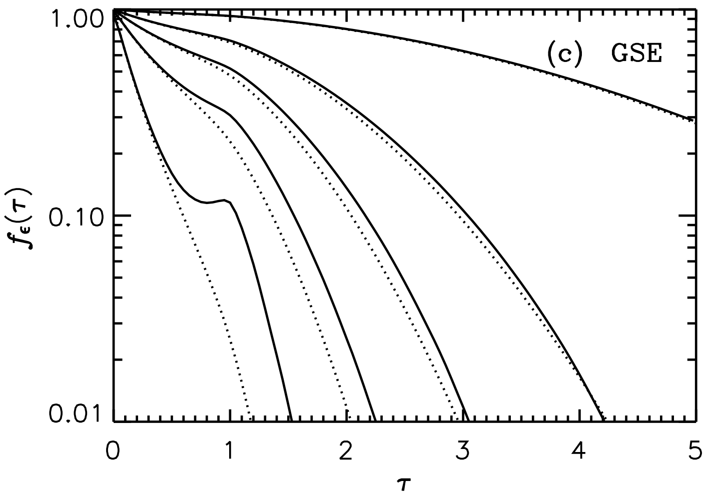
The corresponding calculation for the GOE yields

$$\begin{aligned} f_\epsilon(\tau) &= 2 \int\limits_{\mathrm{Max}(0,\tau-1)}^{\tau} du \int\limits_0^u \frac{v\,dv}{\sqrt{[u^2-v^2][(u+1)^2-v^2]}} \\ &\quad \times \frac{(\tau-u)(1-\tau+u)}{(v^2-\tau^2)^2}[(2u+1)\tau - \tau^2 + v^2]e^{-\frac{1}{2}\epsilon[(2u+1)\tau-\tau^2+v^2]} \end{aligned}$$

Comparison with linear response







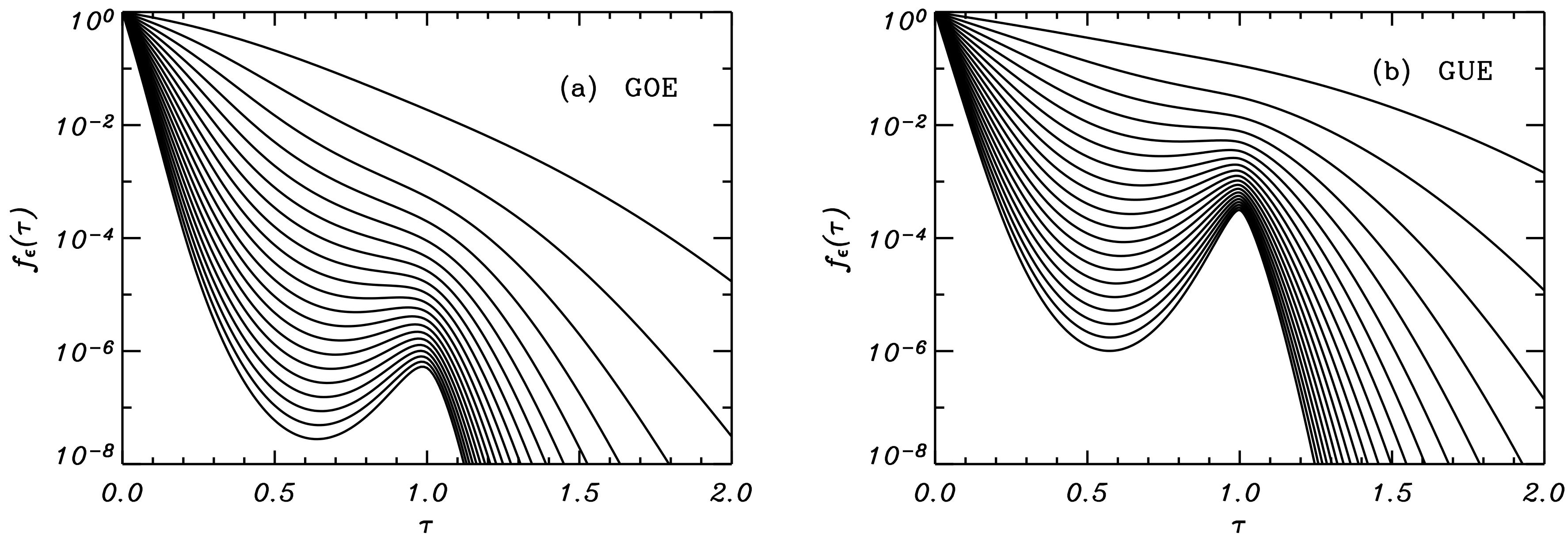
Perturbation strength:

$\epsilon = 0.2, 1, 2, 4, 10$.

Good agreement with linear-response result (dotted) for small ϵ or for small times τ.

(GSE results from numerics.)

The fidelity recovery



Fidelity amplitude $f_\epsilon(\tau)$ for perturbation strengths $\epsilon = 2, 4, \ldots, 40$ for (a) the GOE and (b) the GUE.

Microwave study of the fidelity

Idea:

- Measure spatially resolved pulse-propagation $\psi(\vec{r}, t)$
  (can obtained from the Fourier transform of frequency spectra).
- Compute overlap of pulses for slightly different geometries

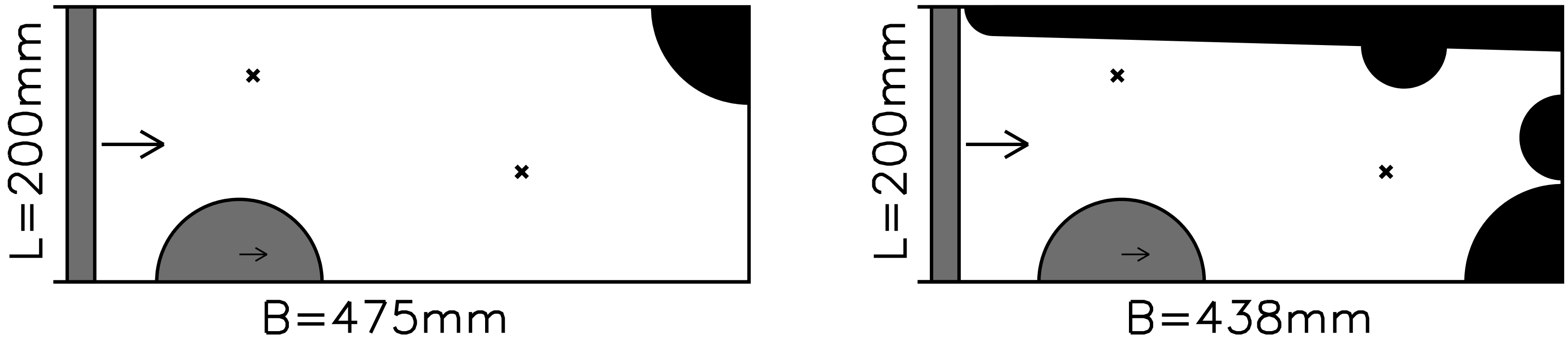
$$f_\lambda(t) = \int \psi(\vec{r}, t)\psi^*_\lambda(\vec{r}, t)\mathrm{d}\vec{r}\,.$$

Problem:

An antenna has to be scanned to measure the wave function
$\Longrightarrow$ additional perturbation!

Fortunately, in chaotic systems things are much easier!

Experimental setup



- Flat microwave resonators ($\nu < c/2d, d$: height)
- Two antennas at fixed positions
- Left wall shifted in 10 steps of $\Delta l = 0.2$ mm.
- Ensemble average by moving half-circle in 15 steps of 2 cm.
- Frequency range: from 3 to 16 GHz ($\approx$ 500 modes)

The scattering fidelity

- Scattering matrix and effective Hamiltonian:

$$S_{ab}(E) = \delta_{ab} - {V^{(a)}}^\dagger \, \frac{1}{E - H_{\rm eff}} \, V^{(b)} \qquad H_{\rm eff} = H_{\rm int} - (\imath/2) \sum_a V^{(a)} \, {V^{(a)}}^\dagger \, ,$$

where V is the coupling matrix and $V_{ja} \sim \psi_j(\vec{r}_a)$.

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  where V is the coupling matrix and $V_{ja} \sim \psi_j(\vec{r}_a)$.

- Consider the Fourier transform of the correlation function

$$\hat{C}[S_{ab}, S_{ab}^{(\lambda)*}](t) = \left\langle \hat{S}_{ab}\,(t)\;\hat{S}_{ab}^{(\lambda)*}\,(-t)\right\rangle\,, \qquad \hat{S}_{ab}\,(t) = \int dE\, e^{\frac{\imath}{\hbar}Et} S_{ab}(E).$$

- Reduces for $\lambda = 0$ to auto-correlation function $\hat{C}[S_{ab}, S_{ab}^{*}](t)$.

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- Consider the Fourier transform of the correlation function

$$\hat{C}[S_{ab}, S_{ab}^{(\lambda)*}](t) = \left\langle \hat{S}_{ab}(t)\, \hat{S}_{ab}^{(\lambda)*}(-t) \right\rangle \,, \qquad \hat{S}_{ab}(t) = \int dE\, e^{\frac{\imath}{\hbar}Et} S_{ab}(E).$$

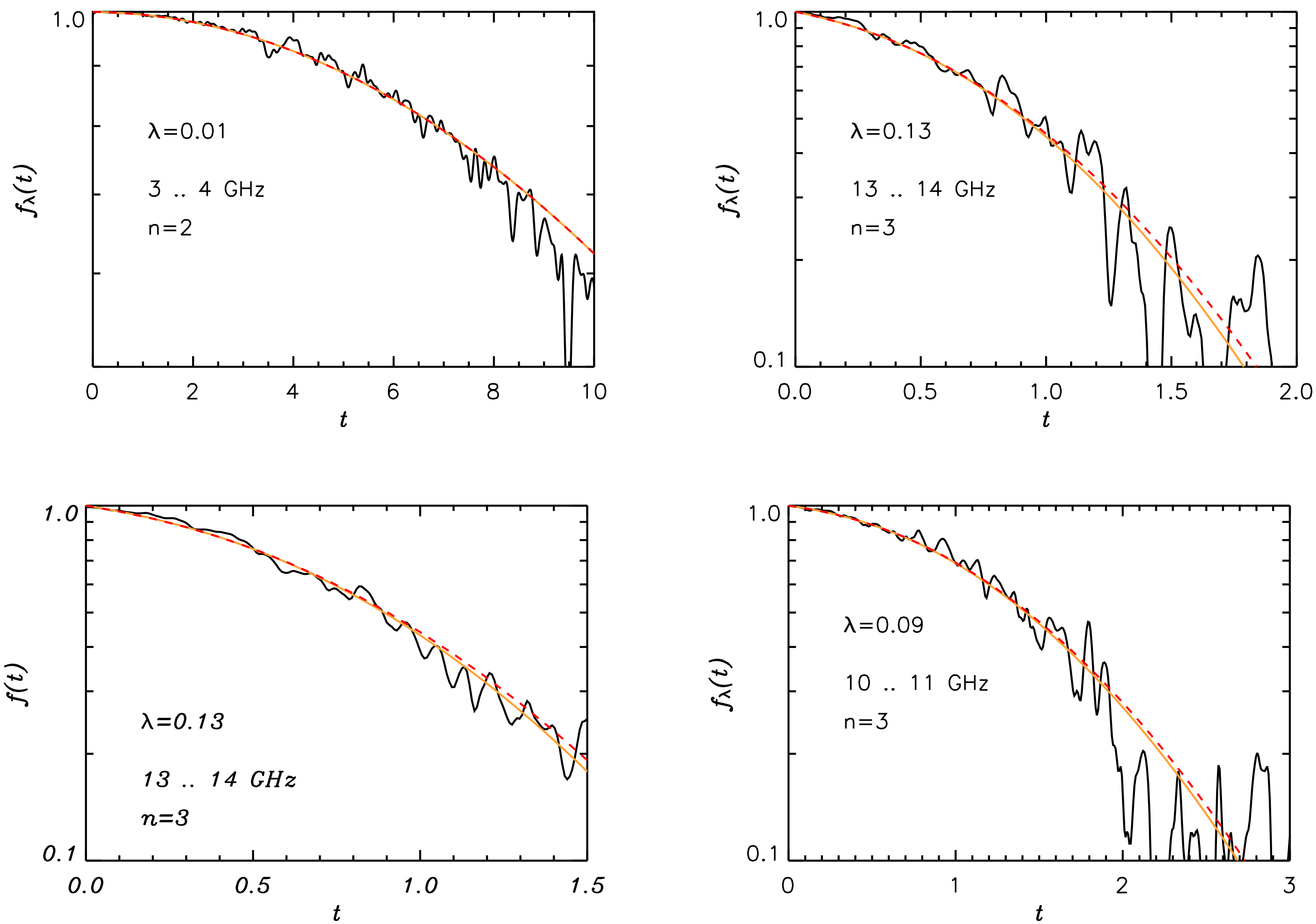
- Reduces for $\lambda = 0$ to auto-correlation function $\hat{C}[S_{ab}, S_{ab}^{*}](t)$.
- Define scattering fidelity via the relation:

$$\hat{C}[S_{ab}, S_{ab}^{(\lambda)*}](t) = f_\lambda^{\rm scat}(t)\, \hat{C}[S_{ab}, S_{ab}^{*}](t)$$

- Reduces in chaotic systems for small perturbations to

$$f_\lambda^{\rm scat}(t) = f_\lambda(t)\,.$$

Experimental fidelity amplitude



- - -: linear response approximation, —: supersymmetry result

Local perturbations

Local fidelity studies

As before: Replace change of Hamiltonian by change of shape!

But there are two alternatives:

- global variation, e. g by change of one length:

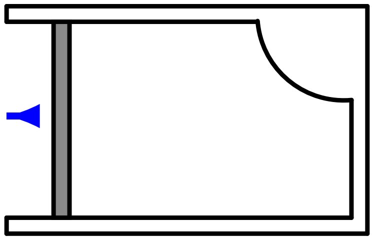
  wave functions are completely changed already for moderate variations (Schäfer *et al.* 2005)

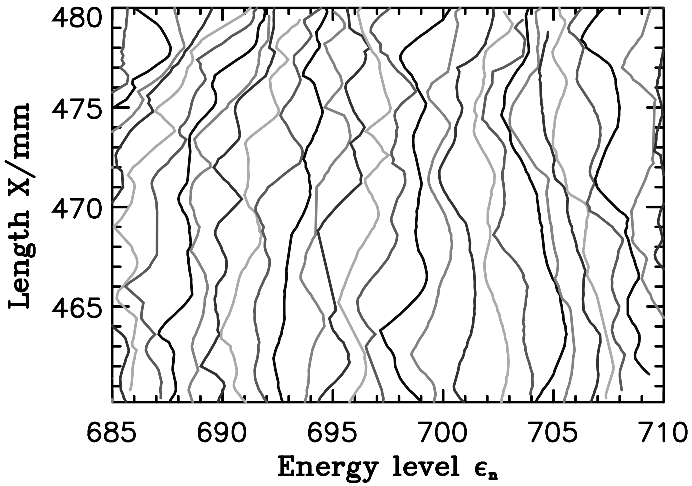
- local variation, e. g. by the change of a scatterer position:

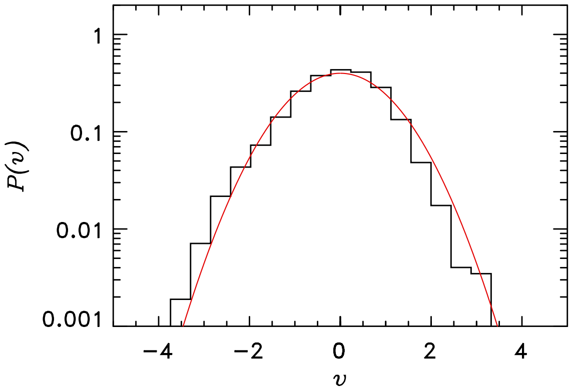
  wave functions are changed only over a distance of some wave lengths

Previous level dynamics studies (Barth *et al.* 1999) suggest that there should be a qualitatively different behavior!

Global versus local parameter variation

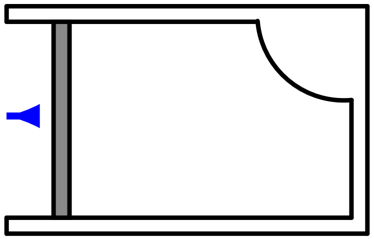


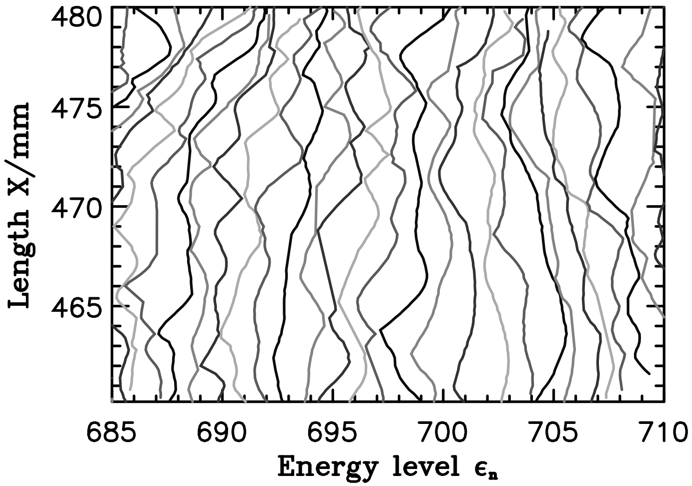


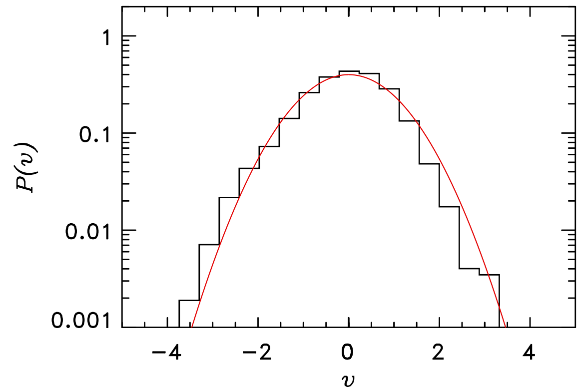


Global variation: Gaussian velocity distribution

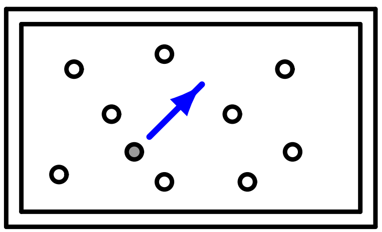
Global versus local parameter variation

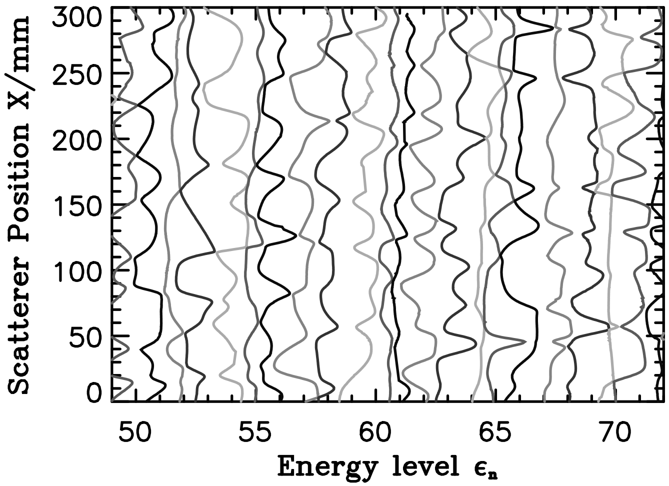


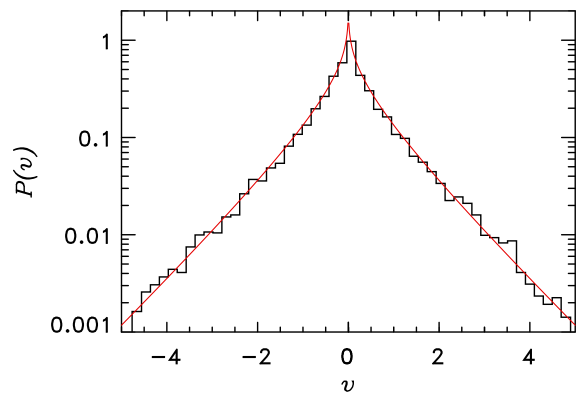




Global variation: Gaussian velocity distribution

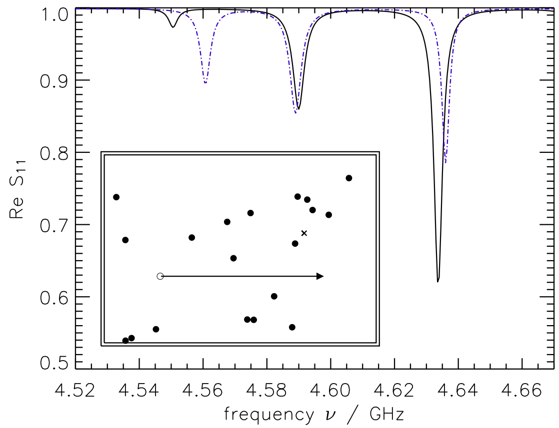






Local variation: modified Bessel velocity distribution

Local perturbations



Scatterer shifts eigenenergies by

$$\Delta E_n \sim |\psi_n(r)|^2$$

$\psi_n(r)$: wave function at the perturber position

It follows for the Hamiltonian

$$H_{nm}(r) = \delta_{nm}(E_n^0 + \alpha|\psi_n(r)|^2),$$

α: scatterer strength

Valid for small shifts only! We never leave the perturbative regime $\Longrightarrow$

$$f(t) = \left\langle e^{2\pi\imath\alpha t(|\psi_1|^2 - |\psi_2|^2)} \right\rangle$$

ψ_1, ψ_2: wave function before and after the shift

The Gaussian average

Random plane wave assumption (Berry 1977) yields

$$f(t) = \frac{\sqrt{|K|}}{2\pi} \iint d\psi_1 d\psi_2 e^{2\pi\imath\alpha(|\psi_1|^2 - |\psi_2|^2)} e^{-\frac{1}{2}(\psi_1,\psi_2)K(\psi_1,\psi_2)^T}$$

where

$$K^{-1} = \begin{pmatrix} \langle\psi_1\psi_1\rangle & \langle\psi_1\psi_2\rangle \\ \langle\psi_2\psi_1\rangle & \langle\psi_2\psi_2\rangle \end{pmatrix} \qquad \langle\psi_i\psi_j\rangle = \frac{1}{A} J_0(k|r_i - r_j|)$$

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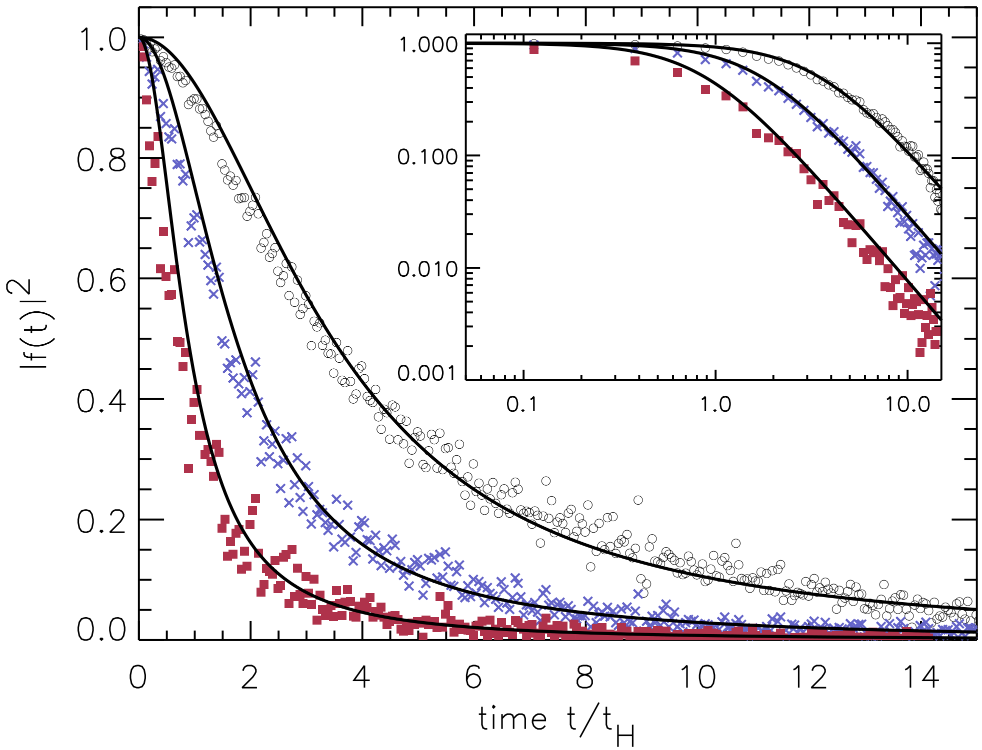
It follows

$$f(t) = \left[1 + (\lambda t)^2\right]^{-\frac{1}{2}} \qquad \lambda = \frac{4\pi\alpha}{A}\sqrt{1 - J_0^2(k|\Delta r|)}$$

Algebraic long-time decay

$$f(t) \sim t^{-1}$$

Results



Fidelity decay for three different perturber shifts $|\Delta r| = 1, 2, 4\,\mathrm{mm}$

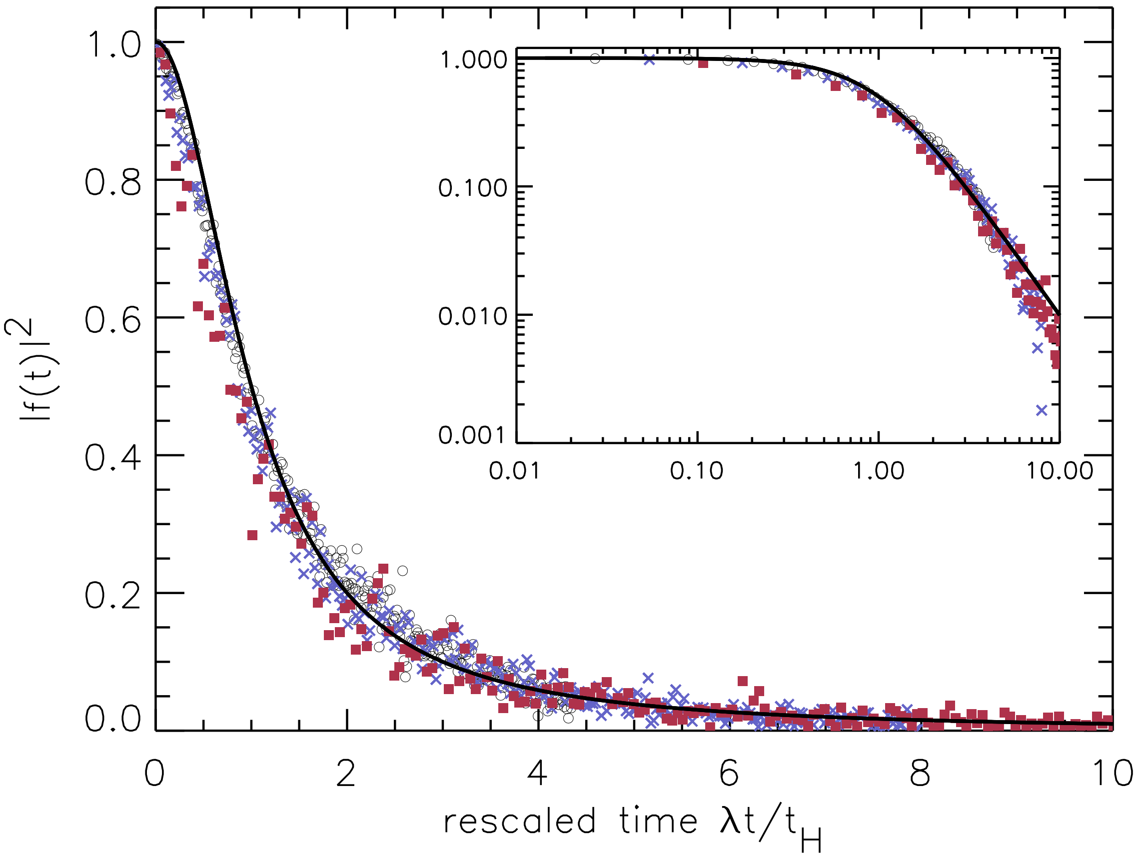
$$f(t) = \left[1 + (\lambda t)^2\right]^{-\frac{1}{2}}$$

$$\lambda = \frac{4\pi\alpha}{A}\sqrt{1 - J_0^2(k|\Delta r|)}$$

No fit applied!

The only free parameter α has been taken from the variance of the level velocities.

Scaling behavior



On a rescaled time axis λt all curves coincide!

Supersymmetries for pedestrians

Idea

Rewrite expression for fidelity

$$
\begin{aligned}
f_\lambda(\tau) &= \langle \psi_0 \,| e^{2\pi\imath H_0 \tau} e^{-2\pi\imath (H_0+\lambda V)\tau} |\, \psi_0 \rangle \\
&= \frac{1}{N} \mathrm{Tr} \left[e^{2\pi\imath H_0 \tau} e^{-2\pi\imath (H_0+\lambda V)\tau} \right]
\end{aligned}
$$

such that the dependencies on the matrix elements factorize.

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$$

such that the dependencies on the matrix elements factorize.

1. Step:

Write the average of the fidelity amplitude as a Fourier transform,

$$\langle f_\lambda(\tau)\rangle = \int dE_1\, dE_2 e^{2\pi\imath(E_1-E_2)\tau}\, R_\lambda\left(E_1, E_2\right)$$

where

$$R_\lambda\left(E_1, E_2\right) \sim \frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{E_{1+}-H_0-\lambda V}\,\frac{1}{E_{2-}-H_0}\right)\right\rangle$$

Idea (*cont.*)

2. Step:

Express A^{-1}_{nm}, where $A = E_{1+} - H_0 - \lambda V$ or $E_{2-} - H_0$ in terms of determinants

$$\left(A^{-1}\right)_{nm} = \frac{1}{|A|}\frac{\partial |A|}{\partial A_{nm}}$$

Idea (*cont.*)

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Express A^{-1}_{nm}, where $A = E_{1+} - H_0 - \lambda V$ or $E_{2-} - H_0$ in terms of determinants

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3. Step:

Write $|A|^{-1}$ as a Fresnel integral

$$\prod_i \left(\int dx_i^* dx_i\right) \exp\left(\imath \sum_{ij} x_i^* A_{ij} x_j\right) = \frac{(2\pi\imath)^N}{|A|}\,.$$

Idea (*cont.*)

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But how to treat $|A|$??

Anticommuting variables

These are sets $\{\xi_n\}$ obeying

$$\xi_i \xi_j + \xi_j \xi_i = 0 \,.$$

Special case: $\xi^2 = 0$, i. e., functions of anticommuting variables are constant or linear!

Integrals over anticommuting variables are defined formally as

$$\int d\xi = 0 \,, \qquad \int \xi d\xi = \frac{1}{\sqrt{2\pi}} \,.$$

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3. Step:

It follows

$$\prod_i \left(\int d\xi_i^* d\xi_i\right) \exp\left(\imath \sum_{ij} \xi_i^* A_{ij} \xi_j\right) = \frac{|A|}{(2\pi\imath)^N}\,.$$

Supersymmetry

Collecting the results:

$$\prod_i \left(\int dx_i^* dx_i\right) \exp\left(\imath \sum_{ij} x_i^* A_{ij} x_j\right) = \frac{(2\pi\imath)^N}{|A|}\,.$$

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Symmetric in the commuting and the anticommuting variables!

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Symmetric in the commuting and the anticommuting variables!

This allows to express inverses of matrices in terms of superintegrals:

$$\left(A^{-1}\right)_{ij} = \frac{1}{|A|}\frac{\partial |A|}{\partial A_{ji}} == \int d[x] \xi_j^* \xi_i \exp\left[\imath \sum_{ij} \left(x_i^* A_{ij} x_j + \xi_i^* A_{ij} \xi_j\right)\right]$$

The Gaussian average

Application to $R_\lambda\left(E_1,E_2\right)$ yields

$$R_\lambda\left(E_1,E_2\right) \sim \frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{E_{1+}-H_0-\lambda V}\,\frac{1}{E_{2-}-H_0}\right)\right\rangle$$

$$= \quad \frac{1}{N}\int d[x]\,d[y]\sum_{n,m}\xi_n^*\xi_m\eta_m^*\eta_n e^{-\imath\left[E_1\mathbf{x}^\dagger\mathbf{x}-E_2\mathbf{y}^\dagger\mathbf{y}\right]}$$

$$\times\left\langle e^{\imath\left[\mathbf{x}^\dagger\mathbf{H}_0\mathbf{x}-\mathbf{y}^\dagger\mathbf{H}_0\mathbf{y}\right]}\right\rangle\left\langle e^{\imath\lambda\mathbf{x}^\dagger\mathbf{V}\mathbf{x}}\right\rangle,$$

where $\mathbf{x}=\left(x_1,\xi_1,\ldots,x_N,\xi_N\right)^T$, $\mathbf{y}=\left(y_1,\eta_1,\ldots,y_N,\eta_N\right)^T$.

Now the Gaussian average over the matrix elements of H_0 and H_1 can be trivially performed!

The Gaussian average

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$$R_\lambda\left(E_1,E_2\right) \sim \frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{E_{1+}-H_0-\lambda V}\,\frac{1}{E_{2-}-H_0}\right)\right\rangle$$

$$= \frac{1}{N}\int d[x]\,d[y]\sum_{n,m}\xi_n^*\xi_m\eta_m^*\eta_n e^{-\imath\left[E_1\mathbf{x}^\dagger\mathbf{x}-E_2\mathbf{y}^\dagger\mathbf{y}\right]}$$

$$\times\left\langle e^{\imath\left[\mathbf{x}^\dagger\mathbf{H}_0\mathbf{x}-\mathbf{y}^\dagger\mathbf{H}_0\mathbf{y}\right]}\right\rangle\left\langle e^{\imath\lambda\mathbf{x}^\dagger\mathbf{V}\mathbf{x}}\right\rangle\,,$$

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Now the Gaussian average over the matrix elements of H_0 and H_1 can be trivially performed!

However: The calculation of the superintegrals is highly non-trivial!

Summary

- Supersymmetry techniques yield exact result for global perturbations
- Local perturbations can be treated in terms of the random plane wave model

- For global perturbations there is a Gaussian or exponential long-time decay
- In contrast the fidelity decays algebraically $\sim t^{-1}$ for local perturbations
- Most perturbations in real systems are short-ranged (e. g. diffusive jumps or mutual spin-flips)
- Possible (positive) implications for quantum computing

Thanks!

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Talks: http://www.physik.uni-marburg.de/qchaoszip/Maribor2008.zip

Papers: http://www.physik.uni-marburg.de/qchaos