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Exact analysis of the adiabatic invariants in time-dependent harmonic oscillator

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CONTENTS

0. PREVIEW

• **adiabatic invariants** denoted by I: long history, important applications, classically and quantally, but rarely rigorous

• they are conserved under very slow changes over an time interval of length T

• definition: **adiabatic parameter** $\epsilon = \frac{1}{T}$: **the ideal adiabatic limit:** $\epsilon \to 0$

• 1D harmonic oscillator $\ddot{q} + \omega^2(t)q = 0$: **general** $\omega(t)$

• if $\epsilon \to 0$: it is known (Einstein 1911): $I = E(t)/\omega(t)$

• define an initial ensemble of phase points with sharp energy E_0:
the uniform canonical ensemble of initial conditions

• distribution of final energy $P(E_1)$ after time $t = T$: **universal distribution**

• $P(E_1)$ is fully determined by the first moment $\bar{E}_1$

• $\mu^2 = \frac{E_0^2}{2}\left((\frac{\bar{E}_1}{E_0})^2 - (\frac{\omega_1}{\omega_0})^2\right)$ **for any** $\omega(t)$

- all higher even moments are powers of μ^2, whilst the odd ones are zero
- the distribution is: $P(E_1) = \frac{1}{\pi\sqrt{2\mu^2 - x^2}}$, where $x = E_1 - \bar{E}_1$
- $T = \infty$ or $\epsilon = 0$: **ideal adiabaticity:** $\mu^2 = \frac{E_0^2}{2}\left((\frac{\bar{E}_1}{E_0})^2 - (\frac{\omega_1}{\omega_0})^2\right) = 0$

and $\bar{E}_1 = \omega_1 E_0/\omega_0$ or $I = E_1/\omega_1 = E_0/\omega_0$

- finite T: calculate $\bar{E}_1$ and μ^2 in general case by exact WKB-theory to all orders
- prove: if $\omega(t)$ is of class $\mathcal{C}^m$ then: $\mu \propto T^{-(m+1)}$, or $\mu^2 \propto \epsilon^{2(m+1)}$
- if $\omega(t)$ analytic then exponential law: $\mu \propto \exp(-\alpha T)$ or $\mu \propto \exp(-\alpha/\epsilon)$
- **distribution** $P(E_1)$ **is universal** (independent of $\omega(t)$) for uniform canonical ensembles of initial conditions.

M. Robnik and V. Romanovski 2006 *J.Phys.A: Math.Gen.* **39** L35-L41
M. Robnik and V. Romanovski 2006 *Open Systems and Information Dynamics* **13** No.2, 197-222
M. Robnik, V. Romanovski, H.-J. Stöckmann 2006 *J.Phys.A: Math.Gen.* **39** L551.
A. V. Kuzmin and M. Robnik 2007 *Rep. on Math. Phys.* **60.1** 69

1. Introduction

Hamilton systems: Phase space (q, p)

Phase flow: $(q_0, p_0) \rightarrow (q_1, p_1)$

Hamilton function $H = H(q, p, t)$

Hamilton equations: $\dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q}$

Energy evolution: $\dot{E} = \frac{dE}{dt} = \frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}}_{=0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

Therefore: The energy E is constant only when $\frac{\partial H}{\partial t} = 0$ (autonomous systems)

Liouville theorem: Phase space volume is *always* preserved: phase space flow velocity vector field has zero divergence ("incompressible flow")

In general, in nonautonomous Hamilton systems, the energy $E = E(t) = H(t)$ *changes with time.*

But, if the changing of the parameter is very slow, on the typical time scale T, there might be a quantity I, a function of the said parameter, of the energy E and of other dynamical quantities, which is approximately conserved.

It might be even exactly conserved if $T \to \infty$, i.e. if the variation is **infinitely slow**, to which case we refer as **the ideal adiabatic variation**.

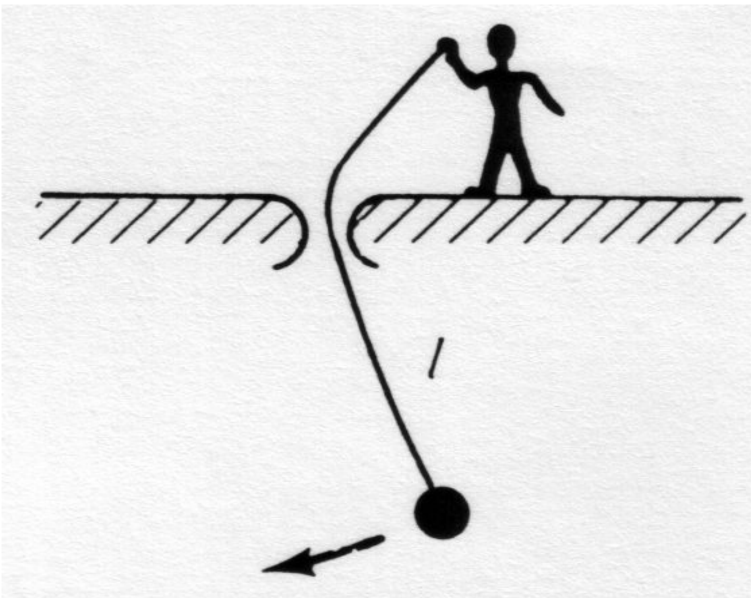
Such a conserved quantity is called **adiabatic invariant**, and it plays an important role in the dynamical analysis of a long-time evolution of *nonautonomous Hamilton systems*.

The theory of adiabatic invariants is aimed at finding the adiabatic invariants I and analyzing the error of its preservation at finite T. Namely, the statement of exactness of I is asymptotic in the sense that the conservation is exact in the limit $T \to \infty$, whilst for finite T we see the deviation $\Delta I = I_f - I_i$ of final value of I_f from its initial value I_i and would like to calculate ΔI. Thus for finite T the final values of I will have some distribution with nonvanishing variance.

In other words, if we start with different initial conditions but at the same fixed and sharply defined initial energy E_0, we observe a distribution of the final energies $P(E_1)$ which has a nonvanishing variance μ^2. We study this distribution function $P(E_1)$.

One-dimensional harmonic oscillator: $\ddot{q} + \omega^2(t)q = 0$

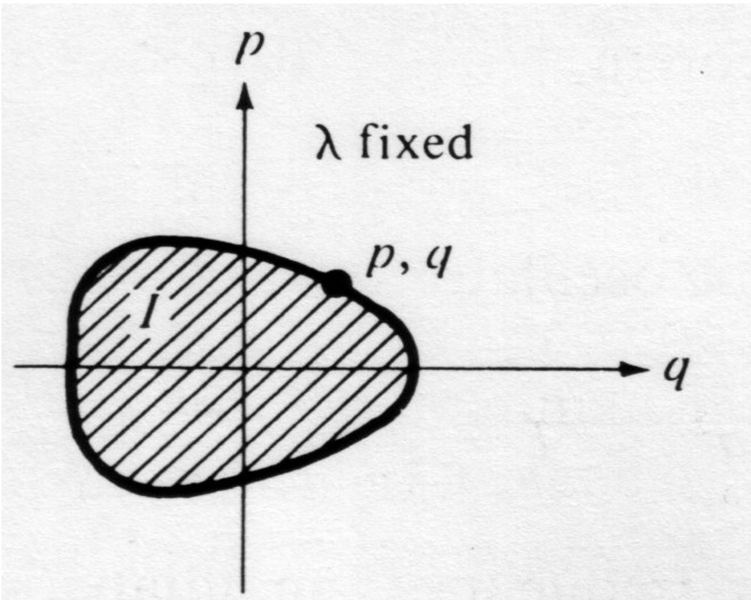
Example: small oscillations of a mathematical pendulum:
$\omega^2(t) = g/l(t)$, where g is the gravitational acceleration and $l(t) =$ the length of the pendulum at time t



It is known since Lorentz (a lecture at the Solway conference 1911) and Einstein (a paper published in 1911): The adiabatic invariant is $I = E(t)/\omega(t)$

This implies: $E(t) = I\omega(t) = I\sqrt{\frac{g}{l(t)}} \propto \frac{1}{\sqrt{l(t)}}$

Please observe: $2\pi I$ is exactly the area in the phase plane (q, p) enclosed by the energy contour of constant E.

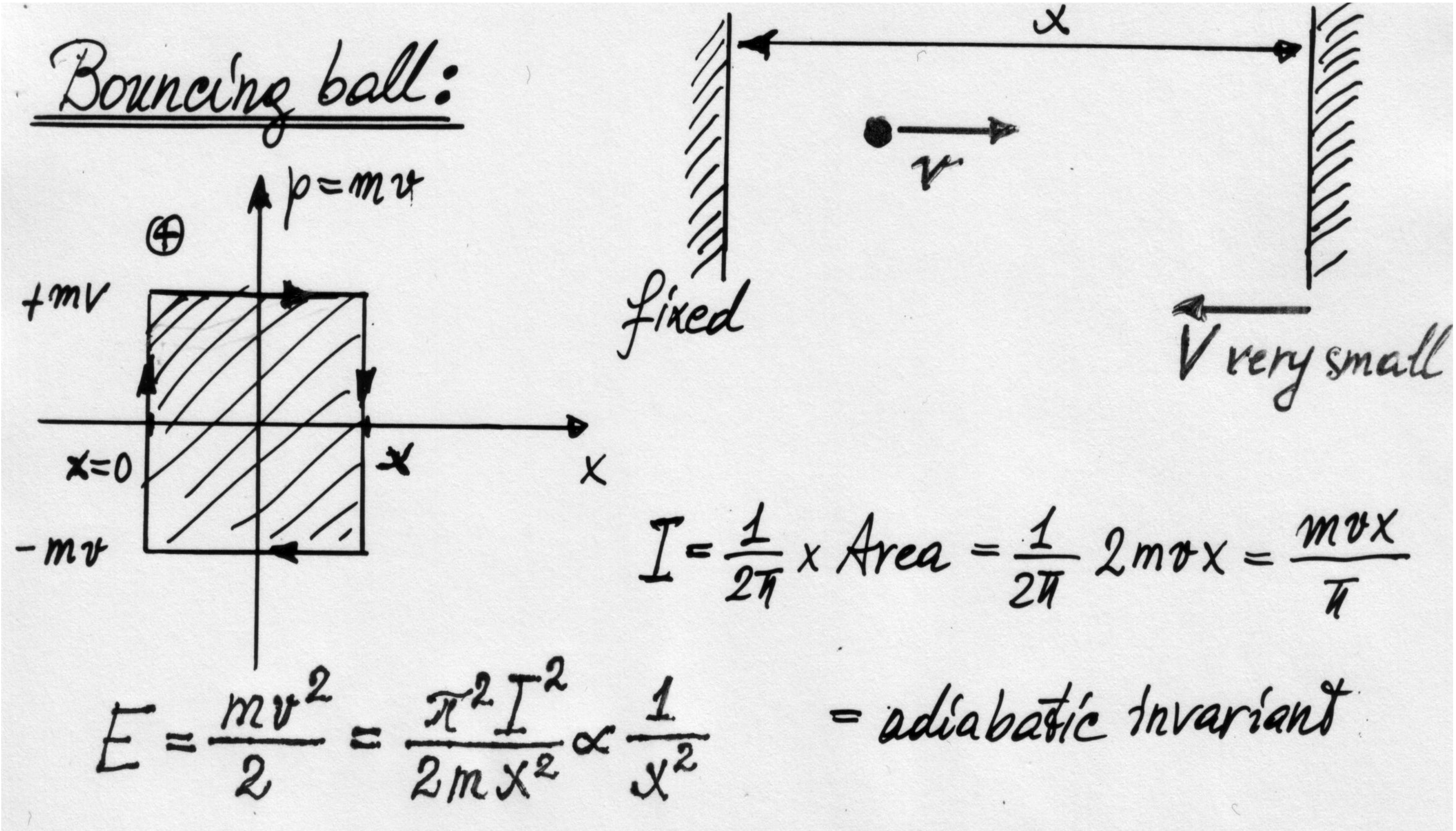


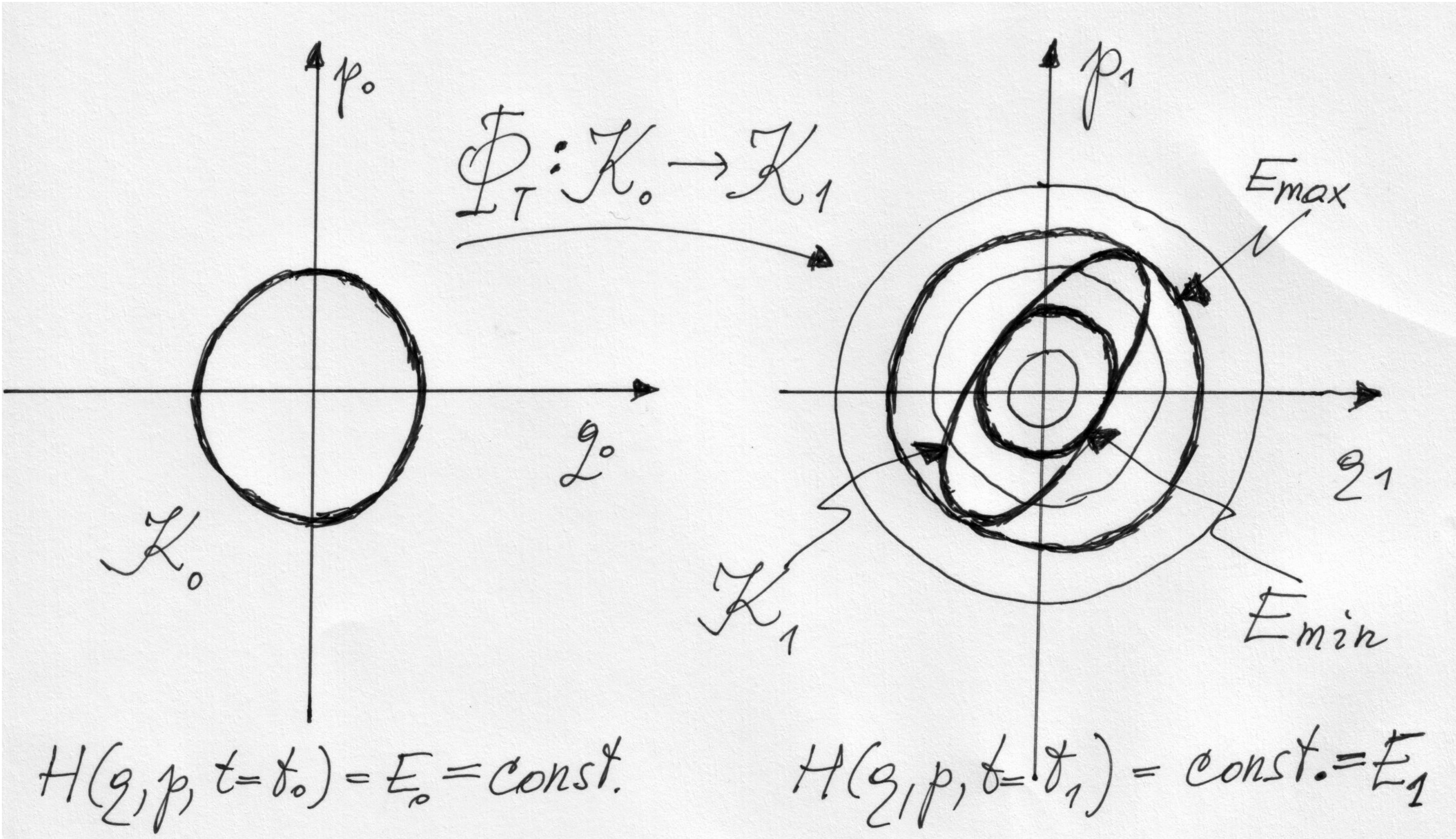
Indeed, in a general 1-dim system with $\omega(t) \neq 0$, the adiabatic invariant I is rigorously equal to

$$I = \frac{1}{2\pi} \oint_{E=H(q,p,t)} p.dq \qquad (1)$$

for 1D harmonic oscillator: $I = E(t)/\omega(t) = const.$ if $T = \infty$

Another elementary example: bouncing ball between two moving planes




p_0
$\Phi_T : K_0 \to K_1$
q_0
K_0
p_1
E_{max}
q_1
K_1
E_{min}
$H(q,p,t=t_0) = E_0 = const.$
$H(q,p,t=t_1) = const. = E_1$

2. Transition map and general exact considerations

The Hamilton function: $H = H(q,p,t) = \frac{p^2}{2M} + \frac{1}{2}M\omega^2(t)q^2$

q, p, M, ω are coordinate, momentum, mass and the frequency.

The numerical value of $H(t)$ is the energy of the system $E(t)$ at time t.

The equation of motion is linear: $\ddot{q} + \omega^2(t)q = 0$

We define the **transition map**: $\Phi : \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \mapsto \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}$.

It is **a linear area preserving map:** $\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $\mathrm{Det}(\Phi) = \mathrm{ad} - \mathrm{bc} = 1$.

Let $E_0 = H(q_0, p_0, t = t_0)$ be the initial energy and $E_1 = H(q_1, p_1, t = t_1)$ be the final energy, that is,

$$E_1 = \frac{1}{2}\left(\frac{(cq_0 + dp_0)^2}{M} + M\omega_1^2(aq_0 + bp_0)^2\right).$$

We want to study the distribution $P(E_1)$ of final energy E_1.

Define the **uniform canonical ensemble of initial conditions:**

$q_0 = \sqrt{\frac{2E_0}{M\omega_0^2}} \cos\phi,\ p_0 = \sqrt{2ME_0} \sin\phi$, where the action is $I_0 = \frac{E_0}{\omega_0}$

Then we obtain: $E_1 = E_0(\alpha\cos^2\phi + \beta\sin^2\phi + \gamma\sin 2\phi)$

with: $\alpha = \frac{c^2}{M^2\omega_0^2} + a^2\frac{\omega_1^2}{\omega_0^2}, \quad \beta = d^2 + \omega_1^2 M^2 b^2, \quad \gamma = \frac{cd}{M\omega_0} + abM\frac{\omega_1^2}{\omega_0}.$

By definition: The distribution of the initial angle variable ϕ is uniform (constant) and equal to $1/(2\pi)$.

The mean value of E_1: $\bar{E}_1 = \frac{1}{2\pi}\oint E_1 d\phi = \frac{E_0}{2}(\alpha + \beta)$.

$x =_{def} E_1 - \bar{E}_1 = E_0(\delta\cos 2\phi + \gamma\sin 2\phi)$, $\delta = \frac{1}{2}(\alpha - \beta)$.

The variance: $\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2}\left(\delta^2 + \gamma^2\right) = \frac{E_0^2}{2}\left[\left(\frac{\bar{E}_1}{E_0}\right)^2 - \left(\frac{\omega_1}{\omega_0}\right)^2\right]$.

Odd moments: $\overline{(E_1 - \bar{E}_1)^{2m-1}} = 0$ Even m.: $\overline{(E_1 - \bar{E}_1)^{2m}} = (2m-1)!!\mu^{2m}/m!$

If $m \to \infty$: $\to 2^m/\sqrt{\pi m}$ (to compare with Gaussian: $\to 2^m\Gamma(m + 1/2)/\sqrt{\pi}$)

The distribution of final energies E_1: $P(E_1) = \frac{1}{2\pi} \sum_{j=1}^{4} \left| \frac{d\phi}{dE_1} \right|_{\phi=\phi_j(E_1)}$

$P(E_1)$ is distributed on (E_{min}, E_{max}), it is an even function w.r.t. $\bar{E}_1 = (E_{min} + E_{max})/2$, and has $1/\sqrt{x}$ singularity at $E_{min} = \bar{E}_1 - \mu\sqrt{2}$ and $E_{max} = \bar{E}_1 + \mu\sqrt{2}$.

The result: $P(E_1) = \frac{1}{\pi\sqrt{2\mu^2 - x^2}}$, where $x = E_1 - \bar{E}_1$

Another way of deriving $P(E_1)$, employing the characteristic function:

$f(y) = \int_{-\infty}^{\infty} e^{iyx} P(x) dx$

n-th derivative at $y = 0$ is: $f^{(n)}(0) = \int_{-\infty}^{\infty} (ix)^n P(x) dx = i^n \sigma_n$

and σ_n is n-th moment of $P(x)$ (in particular: $\sigma_2 = \mu^2$), namely:
$\sigma_n = \int_{-\infty}^{\infty} x^n P(x) dx$

Taylor expansion $f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} y^n$

gives: $f(y) = \sum_{m=0}^{\infty} \frac{i^{2m} \mu^{2m} (2m-1)!!}{m!(2m)!} y^{2m}$

and taking into account formula $(2m-1)!! = (2m)!/(2^m m!)$, we get

$f(y) = \sum_{m=0}^{\infty} \left(-\frac{\mu^2 y^2}{2} \right)^m \frac{1}{(m!)^2}$

which can be summed and we obtain the Bessel function: $f(y) = J_0(\mu y \sqrt{2})$

Now we only invert the Fourier transform, namely

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx} f(y)dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx} J_0(\sqrt{2}\mu y)dy$$

and get: $P(E_1) = \frac{1}{\pi\sqrt{2\mu^2 - x^2}}$, where $x = E_1 - \bar{E}_1$

This is the normalized $\beta(1/2, 1/2)$ distribution (probability density)
or so-called arcus sinus prob. density.
(We shift the origin of x from 0 to $1/2$ and rescale x).

This distribution is **universal** for the 1D harmonic oscillator,
for the case of *uniform canonical ensembles of initial conditions*.

The variance: $\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2}\left[\left(\frac{\bar{E}_1}{E_0}\right)^2 - \left(\frac{\omega_1}{\omega_0}\right)^2\right] \geq 0$ is positive definite.

Therefore in full generality: $\bar{E}_1 \geq E_0\omega_1/\omega_0$

The final value of the adiabatic invariant (for the average energy!) $\bar{I}_1 = \bar{E}_1/\omega_1$ is always greater or equal to the initial value $I_0 = E_0/\omega_0$.

In other words, the value of the adiabatic invariant at the mean value of the energy never decreases, which is a kind of irreversibility statement.

Moreover, it is conserved only for infinitely slow processes $T = \infty$, which is an ideal adiabatic process, for which $\mu = 0$.

For periodic processes $\omega_1 = \omega_0$ we see that always $\bar{E}_1 \geq E_0$, so the mean energy never decreases.

The other extreme to $T = \infty$ is the instantaneous $(T = 0)$ jump where ω_0 switches to ω_1 discontinuously, whilst q and p remain continuous, and this results in $a = d = 1$ and $b = c = 0$, and then we find

$\bar{E}_1 = \frac{E_0}{2}(\frac{\omega_1^2}{\omega_0^2} + 1), \qquad \mu^2 = \frac{E_0^2}{8}\left[\frac{\omega_1^2}{\omega_0^2} - 1\right]^2$. Later: $\omega_1^2 = 2\omega_0^2$, and $\mu^2/E_0^2 = 1/8$.

The calculation of the transition map:

Consider two linearly independent solutions $\psi_1(t)$ and $\psi_2(t)$ of $\ddot{q} + \omega^2(t)q = 0$ and introduce the matrix

$$\Psi(t) = \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ M\dot{\psi}_1(t) & M\dot{\psi}_2(t) \end{pmatrix}.$$

Consider a solution $\hat{q}(t)$ such that $\hat{q}(t_0) = q_0, \quad \dot{\hat{q}}(t_0) = p_0/M$.

Because ψ_1 and ψ_2 are linearly independent, we can look for $\hat{q}(t)$ in the form

$$\hat{q}(t) = A\psi_1(t) + B\psi_2(t).$$

Then A and B are determined by $\begin{pmatrix} A \\ B \end{pmatrix} = \Psi^{-1}(t_0) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$.

Let $q_1 = \hat{q}(t_1), \quad p_1 = M\dot{\hat{q}}(t_1)$. Then **the transition map** arises as follows:

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \Psi(t_1)\Psi^{-1}(t_0) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \Rightarrow \Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Psi(t_1)\Psi^{-1}(t_0).$$

3. Some exactly solvable special cases: *Linear, harmonic and analytic*

3.1 The linear model: class $\mathcal{C}^0$

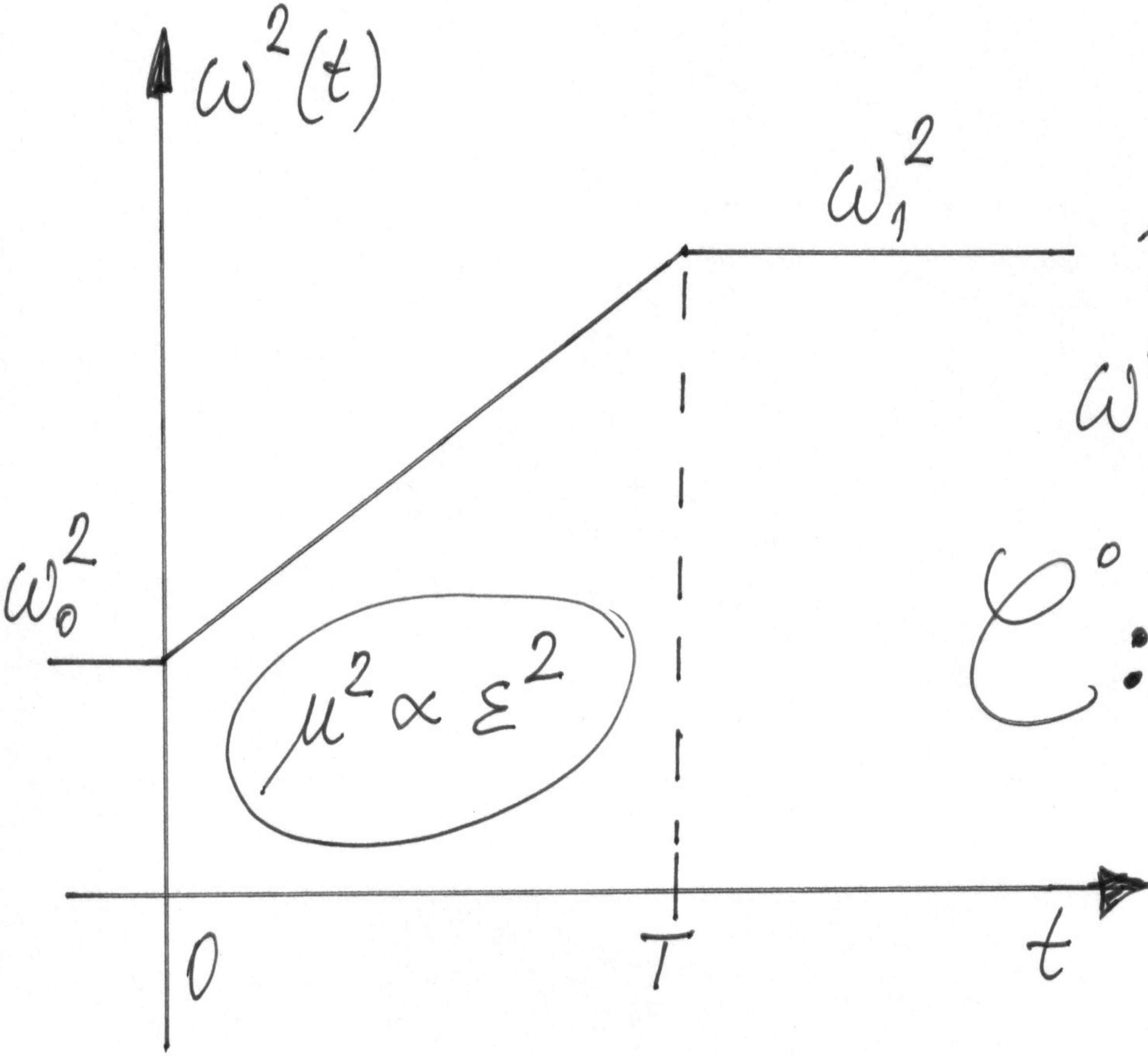
We assume that function $\omega^2(t)$ is a piecewise linear function of the form

$$\omega^2(t) = \begin{cases} \omega_0^2 & \text{if} \quad t \leq 0 \\ \omega_0^2 + \frac{(\omega_1^2 - \omega_0^2)}{T} t & \text{if} \quad 0 < t < T \\ \omega_1^2 & \text{if} \quad t \geq T \end{cases} . \tag{2}$$

Thus $\omega(t)$ has discontinuous first derivative at $t = 0$ and $t = T$, and belongs to the class $\mathcal{C}^0$. Introducing the notation $\tilde{a} = \omega_0^2,\ \tilde{b} = \omega_1^2 - \omega_0^2$ we obtain that on the interval $(0, T)$ the equation has the form

$$\ddot{q} + (\tilde{a} + \frac{\tilde{b}t}{T})q = 0. \tag{3}$$

Two linear independent solutions are given by the Airy functions:
$\psi_1(t) = Ai(\frac{\tilde{b}t + \tilde{a}T}{\tilde{b}^{2/3}T^{1/3}})$ and $\psi_2(t) = Bi(\frac{\tilde{b}t + \tilde{a}T}{\tilde{b}^{2/3}T^{1/3}})$.



linear model

$$\omega^2(t) = \omega_0^2 + \frac{\omega_1^2 - \omega_0^2}{T} t$$

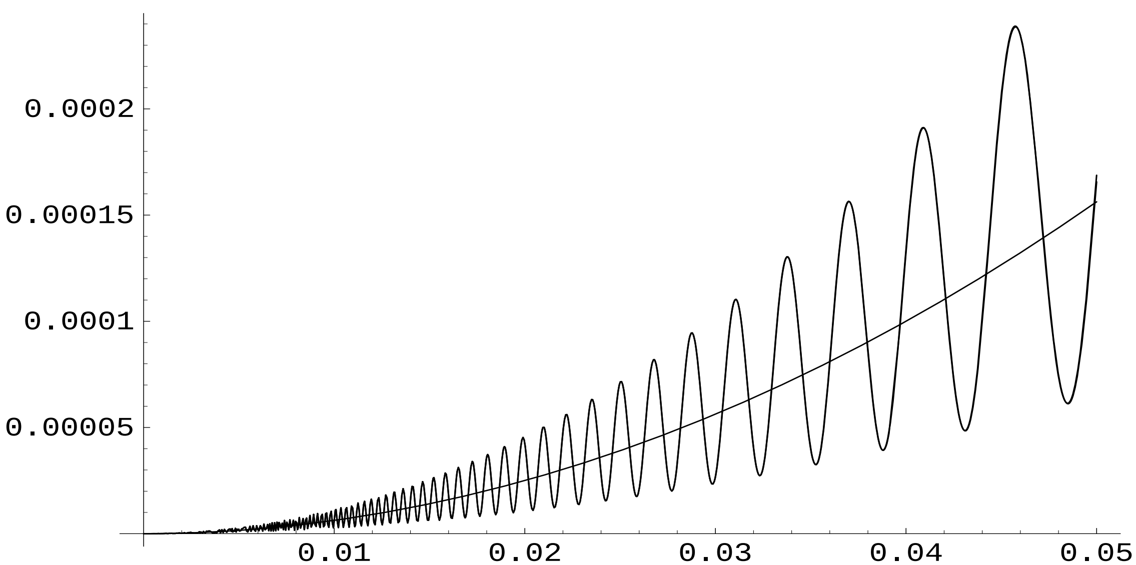
C^0: first derivative discontinuous

$q(t)$: Airy functions

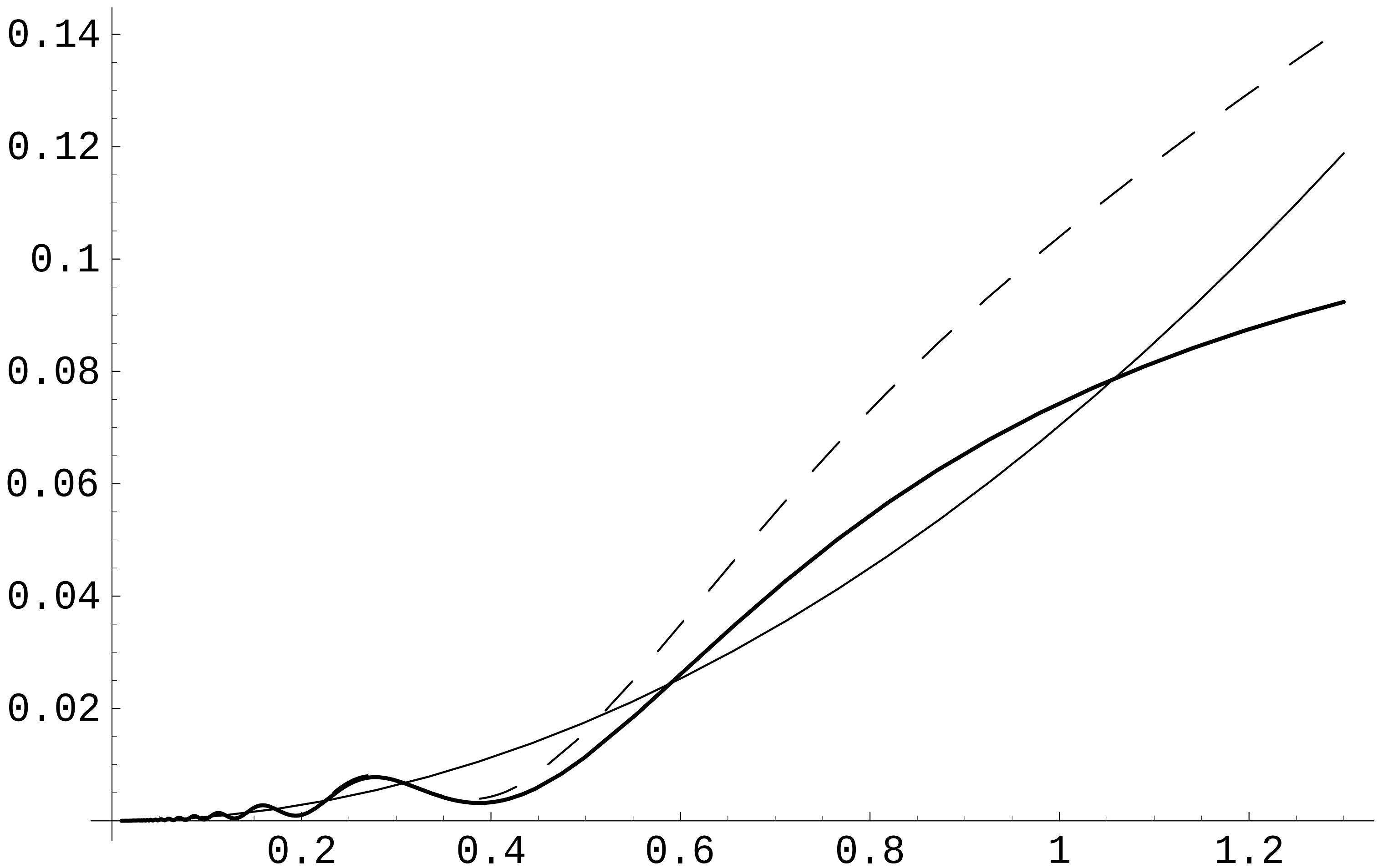
For $\omega_0^2 = 1, \omega_1^2 = 2$, $E_0 = 1$, using the asymptotic expansion of Abramowitz, we obtain the following approximation

$$\overline{(E_1 - \bar{E}_1)^2} \approx \frac{\epsilon^2}{128}\left(9 - 4\sqrt{2}\,\cos(\frac{4 - 8\sqrt{2}}{3\,\epsilon})\right), \tag{4}$$

where we introduce **the adiabatic parameter** $\epsilon = \frac{1}{T}$.



$\overline{(E_1 - \bar{E}_1)^2}$ for $0 < \epsilon < 0.05$; the lines of the exact expression and the asymptotics practically coincide; the non-oscillating thin line is the parabola $y = \frac{9}{128}\epsilon^2$.



$\overline{(E_1 - \bar{E}_1)^2}$ for $0 < \epsilon < 1.2$; the lines of the exact expression (full) and of the asymptotics (dashed) practically coincide for $\epsilon \leq 0.3$; the non-oscillating thin line is the parabola $y = \frac{9}{128}\epsilon^2$. One can show that μ^2 goes to $1/8 = 0.125$ when $\epsilon \to \infty$, which means $T \to 0$, which means the instantaneous jump of $\omega_0 = 1$ to $\omega_1 = \sqrt{2}$.

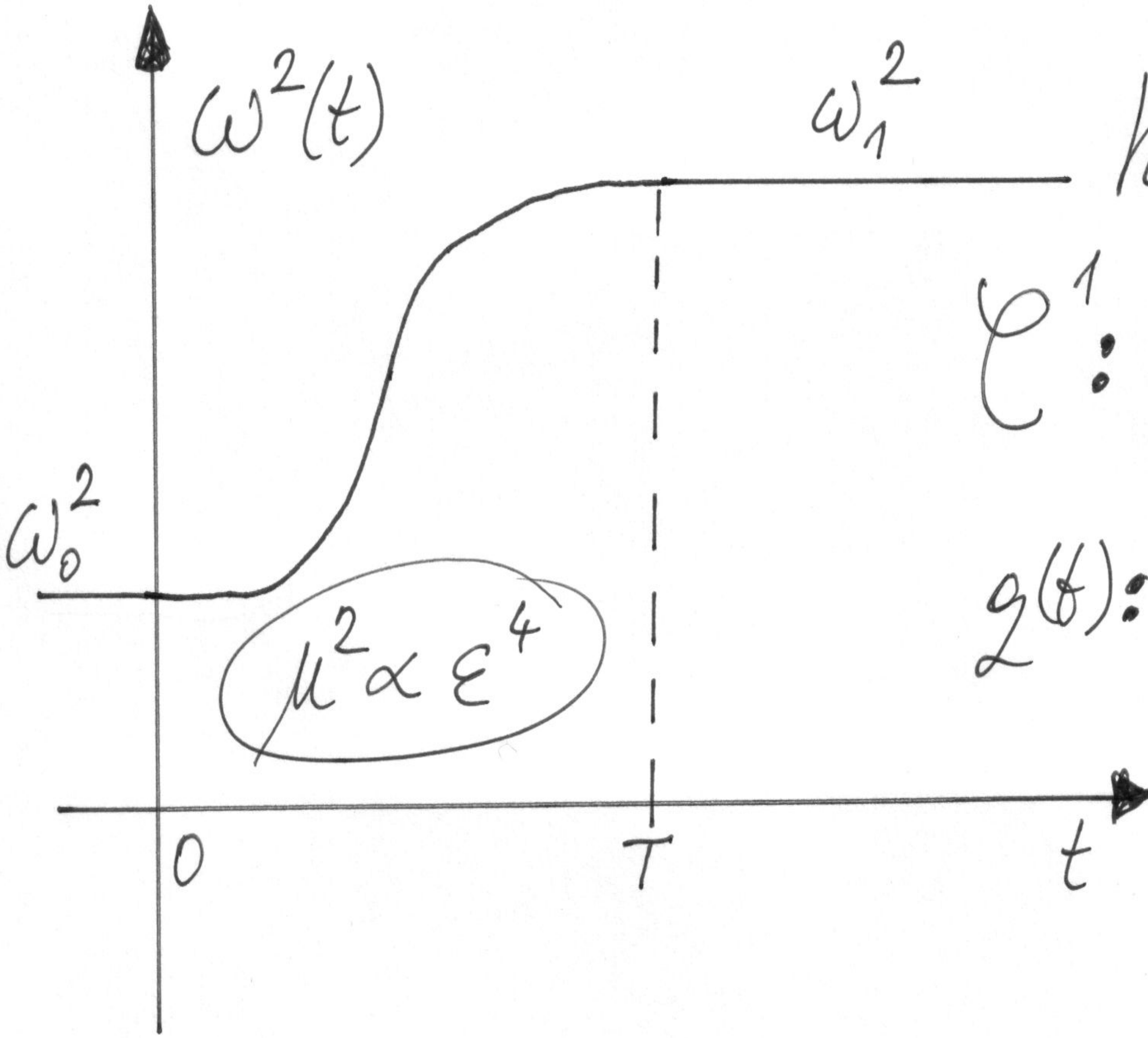
3.2 The harmonic model: class $\mathcal{C}^1$

$$\omega^2(t) = \begin{cases} \tilde{a} & \text{if} \quad t \leq 0 \\ \tilde{b} - (\tilde{b} - \tilde{a}) \cos(\frac{\pi\, t}{T}) & 0 < t < T \\ 2\tilde{b} - \tilde{a} & \text{if} \quad t \geq T \end{cases}, \tag{5}$$

where $\tilde{a} = \omega_0^2,\ 2\tilde{b} - \tilde{a} = \omega_1^2$.

Then the Newton equation has the form

$$\ddot{q} + \left(\tilde{b} - (\tilde{b} - \tilde{a}) \cos\left(\frac{\pi t}{T}\right)\right) q = 0 \tag{6}$$

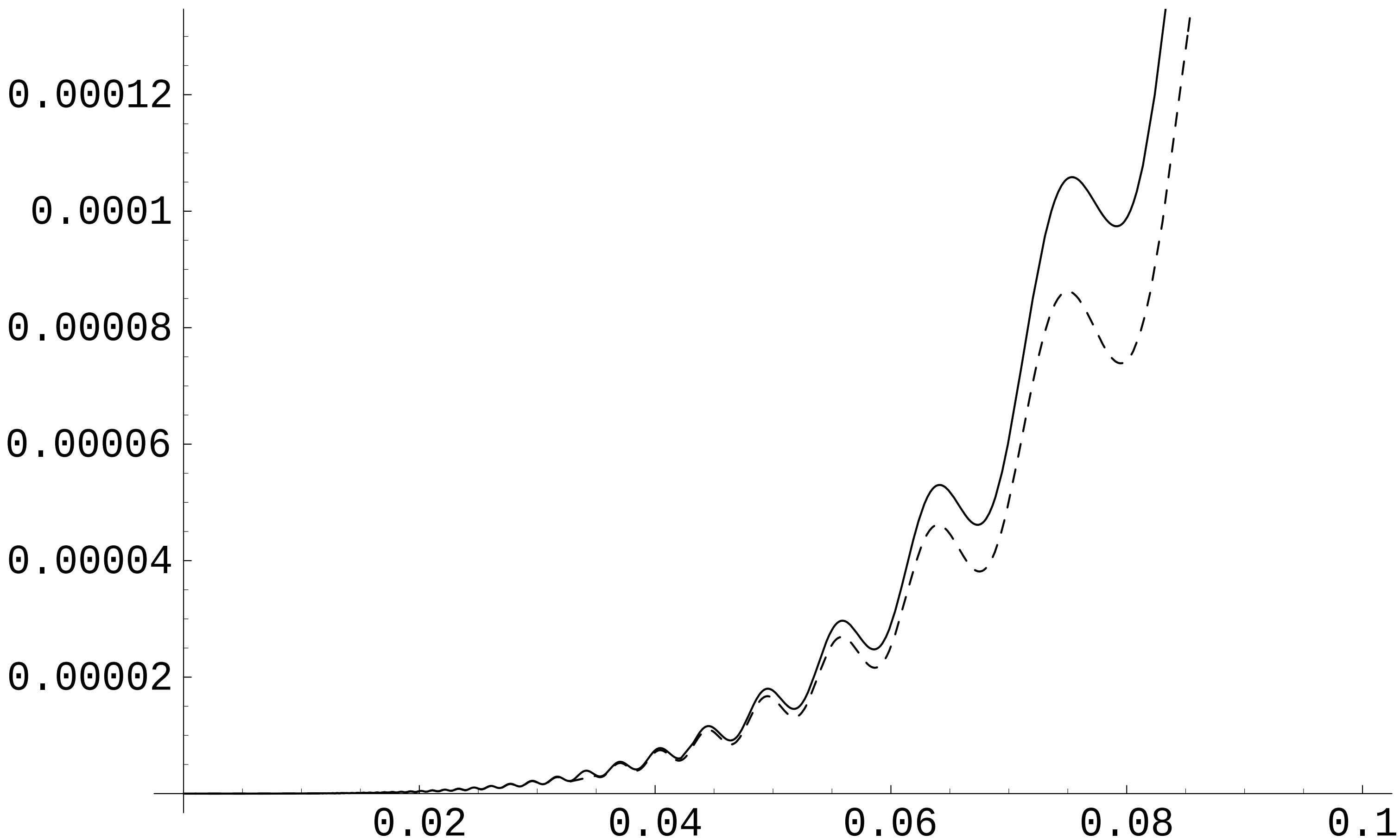


harmonic model

C^1: second derivative discontinuous

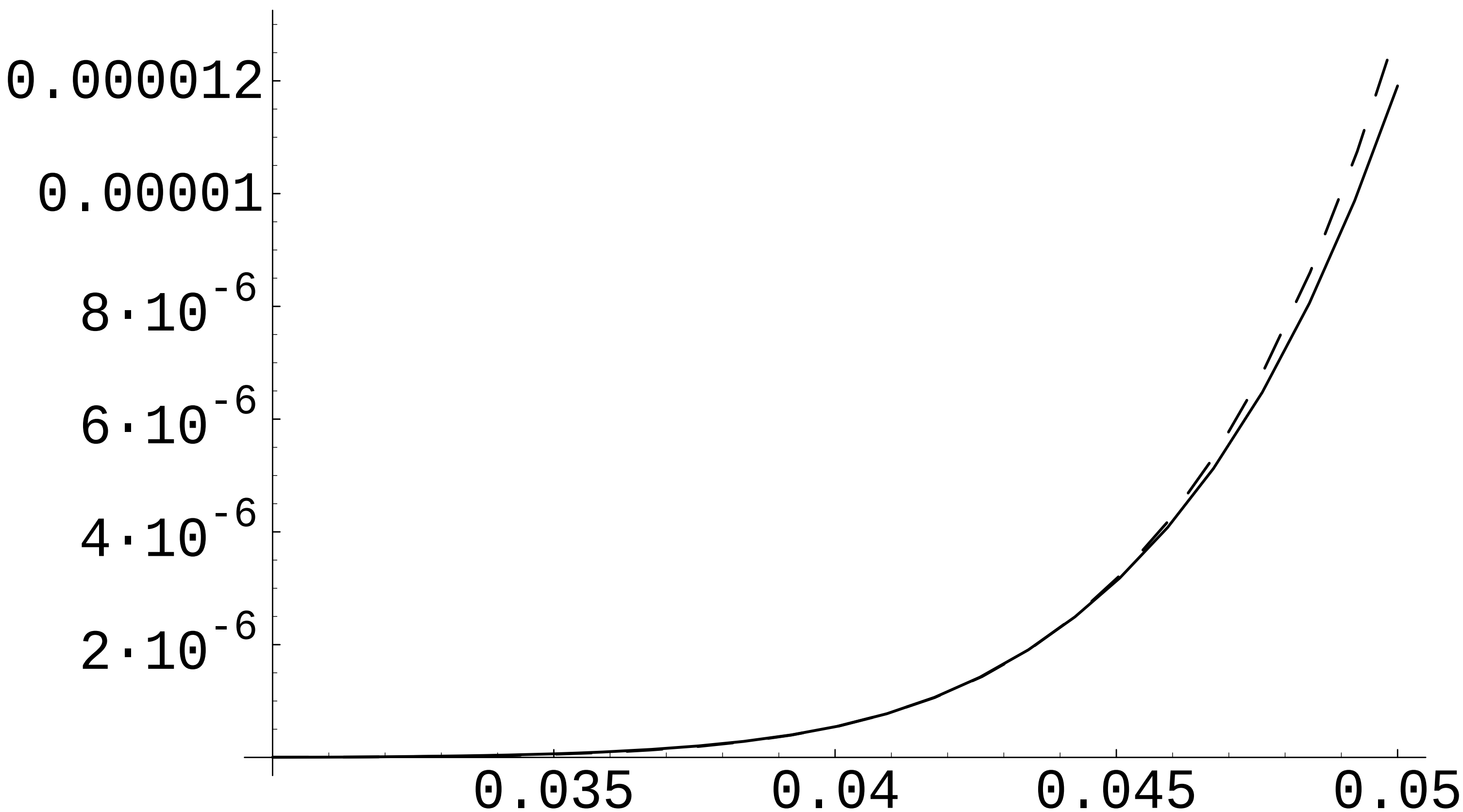
$q(t)$: Mathieu functions

It can be solved in terms of Mathieu functions.

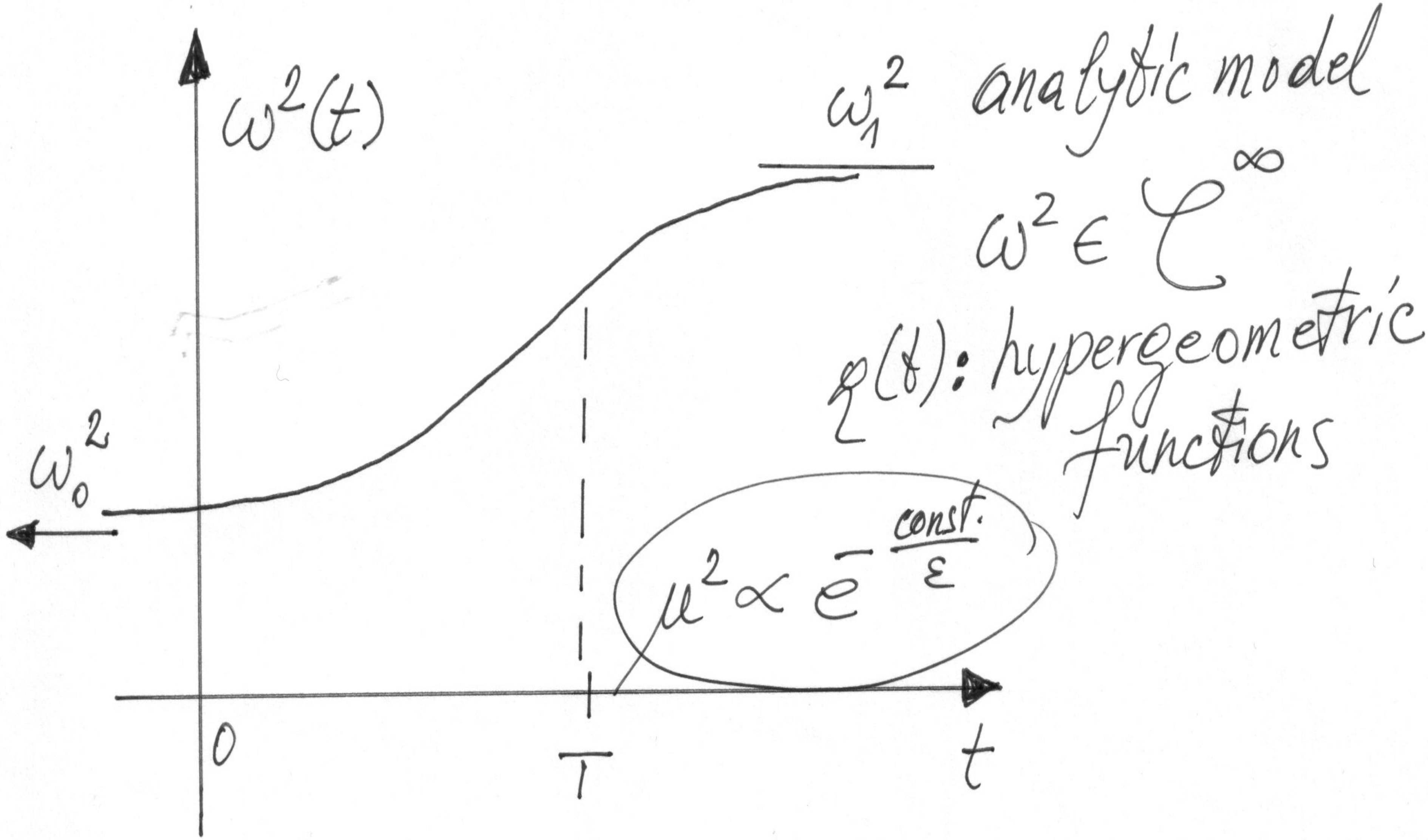


We show, for the harmonic model of subsection 3.2, $\overline{(E_1 - \bar{E}_1)^2}$ for $0 < \epsilon < 0.1$. The exact result is represented by the full line, whilst the dashed curve is the curve $0.056\,\epsilon^4\,\left(41 + 9\,\cos(\frac{2.78}{\epsilon})\right)$, obtained by the WKB method.

The analytic model: class $\mathcal{C}^\infty$: $\omega^2(t) = \frac{1+a\,e^{\alpha t}}{1+e^{\alpha t}}$



The variance $\overline{(E_1 - \bar{E}_1)^2}$ of the energy for the analytic model, for $0.03 < \epsilon < 0.05$. The dashed curve is approximation $y = 4.174e^{-0.634/\epsilon}$.


$\omega^2(t)$
ω_1^2
ω_0^2
0
T
t
analytic model
$\omega^2 \in C^\infty$
$q(t)$: hypergeometric functions
$\mu^2 \propto e^{-\frac{const.}{\varepsilon}}$

4. Application of the WKB method for the calculation of the transition map

4.1 General and exact considerations

Since the adiabatic limit $\epsilon \to 0$ is the asymptotic regime that we would like to understand, the WKB method seems to be the most suitable approach. We are able to work out WKB terms to all orders, and in that sense the theory so far is exact. In practice, however, very often the few leading terms are sufficient, or even just the leading term.

We introduce re-scaled and dimensionless time λ: $\lambda = \epsilon t, \quad \epsilon = 1/T$,

so that $\ddot{q} + \omega^2(t) q = 0$ is transformed to the equation: $\epsilon^2 q''(\lambda) + \omega^2(\lambda) q(\lambda) = 0$.

Let $q_+(\lambda)$ and $q_-(\lambda)$ be two linearly independent solutions: Then

$$\Psi_\lambda = \begin{pmatrix} q_+(\lambda) & q_-(\lambda) \\ \epsilon M q'_+(\lambda) & \epsilon M q'_-(\lambda) \end{pmatrix} \tag{7}$$

and taking into account that $\lambda_0 = \epsilon t_0, \lambda_1 = \epsilon t_1$ we obtain for the matrix the expression

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Psi_\lambda(\lambda_1)\Psi_\lambda^{-1}(\lambda_0). \tag{8}$$

We now use the WKB method in order to obtain the coefficients a, b, c, d of the matrix Φ. To do so, we look for solution in the form $q(\lambda) = w \exp\left\{\frac{1}{\epsilon}\sigma(\lambda)\right\}$

where $\sigma(\lambda)$ is a complex function that satisfies the differential equation

$(\sigma'(\lambda))^2 + \epsilon\sigma''(\lambda) = -\omega^2(\lambda)$

and w is some constant with dimension of length.

The WKB expansion for the phase is $\sigma(\lambda) = \sum_{k=0}^{\infty} \epsilon^k \sigma_k(\lambda)$. Substituting and comparing like powers of ϵ gives the recursion relation

$$\sigma_0'^2 = -\omega^2(\lambda), \qquad \sigma_n' = -\frac{1}{2\sigma_0'}\left(\sum_{k=1}^{n-1} \sigma_k'\sigma_{n-k}' + \sigma_{n-1}''\right). \tag{9}$$

Here we apply our WKB notation and formalism from our work (Robnik and Romanovski 2000, [10]) and we can choose
$\sigma'_{0,+}(\lambda) = \mathrm{i}\omega(\lambda) \quad \text{or} \quad \sigma'_{0,-}(\lambda) = -\mathrm{i}\omega(\lambda)$. That results in two linearly independent solutions given by the WKB expansions with the coefficients

$$\sigma_{0,\pm}(\lambda) = \pm \mathrm{i} \int_{\lambda_0}^{\lambda} \omega(x) dx, \quad \sigma_{1,\pm}(\lambda) = -\frac{1}{2} \log \frac{\omega(\lambda)}{\omega(\lambda_0)}, \tag{10}$$

$$\sigma_{2,\pm} = \pm \frac{\mathrm{i}}{8} \int_{\lambda_0}^{\lambda} \frac{3\,\omega'(x)^2 - 2\,\omega(x)\,\omega''(x)}{\omega(x)^3}\, dx, \ \ldots \tag{11}$$

Since $\omega(\lambda)$ is a real function we deduce that all functions σ'_{2k+1} are real and all functions σ'_{2k} are pure imaginary and $\sigma'_{2k,+} = -\sigma'_{2k,-}, \quad \sigma'_{2k+1,+} = \sigma'_{2k+1,-}$ where $k = 0, 1, 2, \ldots$, and thus we have $\sigma'_+ = A(\lambda) + \mathrm{i}B(\lambda), \quad \sigma'_- = A(\lambda) - \mathrm{i}B(\lambda)$ where $A(\lambda) = \sum_{k=0}^{\infty} \epsilon^{2k+1} \sigma'_{2k+1}(\lambda), \quad B(\lambda) = -\mathrm{i} \sum_{k=0}^{\infty} \epsilon^{2k} \sigma'_{2k,+}(\lambda)$.

Integration of the above equations yields $\sigma_+ = r(\lambda) + \mathrm{i}s(\lambda), \quad \sigma_- = r(\lambda) - \mathrm{i}s(\lambda)$,

where $r(\lambda) = \int_{\lambda_0}^{\lambda} A(x)\, dx, \quad s(\lambda) = \int_{\lambda_0}^{\lambda} B(x)\, dx$.

Below we shall denote $s_1 = s(\lambda_1)$.

To simplify the expressions let us denote $A_0 = A(\lambda_0)$, $A_1 = A(\lambda_1)$, $B_0 = B(\lambda_0)$ and $B_1 = B(\lambda_1)$.

After a long calculation we obtain:

$$
\begin{aligned}
\alpha + \beta = \frac{1}{B_0 B_1} \Bigg[& \sin^2\left(\frac{s_1}{\epsilon}\right)\left(\frac{B_0^2 B_1^2}{\omega_0^2} + \omega_1^2\right) + \cos^2\left(\frac{s_1}{\epsilon}\right)\left(B_0^2 \frac{\omega_1^2}{\omega_0^2} + B_1^2\right) + \\
& \sin^2\left(\frac{s_1}{\epsilon}\right)\left(A_0^2 \frac{\omega_1^2}{\omega_0^2} + \frac{A_0^2 A_1^2}{\omega_0^2} + \frac{2 A_0 A_1 B_0 B_1}{\omega_0^2} + A_1^2\right) + \\
& \cos^2\left(\frac{s_1}{\epsilon}\right)\left(\frac{A_0^2 B_1^2}{\omega_0^2} + \frac{A_1^2 B_0^2}{\omega_0^2} - \frac{2 A_0 A_1 B_0 B_1}{\omega_0^2}\right) + \\
& \sin\left(\frac{s_1}{\epsilon}\right)\cos\left(\frac{s_1}{\epsilon}\right) \times \\
& \left(-2 A_0 B_0 \frac{\omega_1^2}{\omega_0^2} + 2 A_1 B_1 + \frac{2}{\omega_0^2}\left(A_0 A_1 + B_0 B_1\right)\left(A_0 B_1 - A_1 B_0\right)\right)\Bigg].
\end{aligned}
\tag{12}
$$

Solving the WKB recurrence equation for the differential equation:

$\epsilon^2 q''(\lambda) + \omega^2(\lambda) q(\lambda) = 0$

with the **ansatz**: $q(\lambda) = w \exp[\frac{1}{\epsilon}\sigma(\lambda)]$

yielding: $(\sigma'(\lambda))^2 + \epsilon\sigma''(\lambda) = Q(\lambda) = -\omega^2(\lambda)$

and expanding : $\sigma(\lambda) = \sum_{k=0}^{\infty} \epsilon^k \sigma_k(\lambda)$

gives: $\sigma_0'^2 = -\omega^2(\lambda)$, $\qquad \sigma_n' = -\frac{1}{2\sigma_0'}(\sum_{k=1}^{n-1} \sigma_k' \sigma_{n-k}' + \sigma_{n-1}'')$

The solution is (proof by induction):

Following our work (Robnik in Romanovski 2000 J.Phys.A **33** 5093)

Let $M = \cup_{k=1}^{\infty} \mathbf{N}^k$, $\mathbf{N}$ is the set of non-negative integers. We define the map $L : M \to \mathbf{N}$ by

$$L(\nu) = 1 \cdot \nu_1 + 2 \cdot \nu_2 + \ldots + l \cdot \nu_l \qquad (13)$$

and denote by $L(\nu)=m$ the equation

$$L(\nu)=1\cdot\nu_1+2\cdot\nu_2+\ldots+m\cdot\nu_m=m, \tag{14}$$

with $m\in\mathbf{N}$, $\nu\in M$. For a vector $\nu=(\nu_1,\ldots,\nu_l)\in M$ we denote $Q^{(\nu)}=(Q')^{\nu_1}(Q'')^{\nu_2}\ldots(Q^{(l)})^{\nu_l}$, $|\nu|=\nu_1+\ldots+\nu_l$ and let $\nu(i)$ $(i=1,\ldots,l-1)$ be the vector $(\nu_1,\ldots,\nu_i+1,\nu_{i+1}-1,\ldots,\nu_l)$. The functions σ'_m are of the form:

$$\sigma'_m=\sum_{\nu:L(\nu)=m}\frac{U_\nu Q^{m-|\nu|}Q^{(\nu)}}{Q^{\frac{3m-1}{2}}}, \tag{15}$$

where the coefficients U_ν satisfy the recurrence relation

$$U_\nu=\tfrac{1}{2}\textstyle\sum_{\mu,\theta\neq 0,\mu+\theta=\nu}U_\mu U_\theta+\frac{(4-L(\nu)-2|\nu|)U_{(\nu_1-1,\nu_2,\ldots,\nu_l)}}{4}+\sum_{i=1}^{l-1}\frac{(\nu_i+1)U_{\nu(i)}}{2}$$

with $U_{\bar{0}}=-1$ and we put $U_\alpha=0$ if among the coordinates of the vector α there is a negative one.

4.2 Leading asymptotic terms in the power expansion in terms of ϵ

So far the result is exact. Let us consider the first order WKB approximation, which is the generic case, that is

$$A(\lambda) \approx \epsilon\sigma'_{1,+}(\lambda), \quad B(\lambda) \approx \frac{\sigma'_{0,+}(\lambda)}{\mathrm{i}} = \omega(\lambda).$$

Substituting these values of $A(\lambda)$ and $B(\lambda)$ we find, $\overline{E_1} = E_0(\alpha+\beta)/2$:

$$\alpha+\beta = 2\frac{\omega_1}{\omega_0} + \epsilon^2 \left(\frac{\omega_1 \omega'(\lambda_0)^2}{4\omega_0^5} - \frac{\cos\left(\frac{2\int_{\lambda_0}^{\lambda_1}\omega(x)\,dx}{\epsilon}\right)\omega'(\lambda_0)\omega'(\lambda_1)}{2\omega_0^3\omega_1} + \frac{\omega'(\lambda_1)^2}{4\omega_0\omega_1^3} \right) + O(\epsilon^3).$$

$$\frac{\mu^2}{E_0^2} = \frac{\overline{(\Delta E_1)^2}}{E_0^2} = \frac{1}{2}\left[\left(\frac{\bar{E}_1}{E_0}\right)^2 - \left(\frac{\omega_1}{\omega_0}\right)^2\right] =$$

$$\epsilon^2 \left(\frac{\omega_1^2 \omega'(\lambda_0)^2}{8\omega_0^6} - \frac{\cos\left(\frac{2\int_{\lambda_0}^{\lambda_1}\omega(x)\,dx}{\epsilon}\right)\omega'(\lambda_0)\omega'(\lambda_1)}{4\omega_0^4} + \frac{\omega'(\lambda_1)^2}{8\omega_0^2\omega_1^2} \right) + O(\epsilon^3).$$

4.3 Further simplifications of the general formula for the leading terms

The special cases: If all derivatives at λ_0 and λ_1 vanish up to order $(n-1)$

$$\omega'(\lambda_0) = \ldots = \omega^{(n-1)}(\lambda_0) = \omega^{(n-1)}(\lambda_1) = 0,\ \omega^{(n)}(\lambda_0)\omega^{(n)}(\lambda_1) \neq 0.$$

then using our theory from (Robnik and Romanovski 2000,[10]) we find:

$$\frac{\mu^2}{E_0^2} = \frac{\overline{(\Delta E_1)^2}}{E_0^2} = \frac{\epsilon^{2n}}{2^{2n+1}}\left(\frac{\omega_1^2(\omega_0^{(n)})^2}{\omega_0^{2(n+2)}} + \frac{(\omega_1^{(n)})^2}{(\omega_1)^{2n}\omega_0^2} - 2\frac{\omega_0^{(n)}\omega_1^{(n)}}{\omega_0^{n+3}\omega_1^{n-1}}\cos\left(\frac{2s_1}{\epsilon}\right)\right) + O(\epsilon^{2n+1}).$$

In the special case $n=1$ we recover previous formula.

We see:

Theorem:

If $\omega(t)$ is of class $\mathcal{C}^m$, meaning having m-th continuous derivative, then μ^2 is oscillating as $\epsilon \to 0$ but in the mean goes to zero as a power $\mu^2 \propto \epsilon^{2(m+1)}$.

This achievement demonstrates the power of the WKB method.

If $\omega(t)$ is an analytic function on the real time axis $(-\infty, +\infty)$, the decay to zero is oscillating and on the average is exponential $\propto \exp(-const/\epsilon)$ or $\propto \exp(-const.T)$

Let us now summarize our results for the variance μ^2 as a function of $\omega(t)$ embodied in the exact general formulae above.

If $\omega(t)$ is analytic between t_0 and $t_1 \geq t_0$ then the main equation above applies, and as we see μ^2 is dominated by the switch-on and switch-off events at t_0 and t_1, respectively. However, the smaller the jump in the derivatives of $\omega(t)$ at the two points, the smaller will be the power law contribution. Indeed, if t_0 and t_1 go to $-\infty$ and $+\infty$, respectively, and if $\omega(t)$ is analytic on the entire interval, then the behaviour is exponential at sufficiently large $\epsilon \geq \epsilon_c$, but a power law at small $\epsilon \leq \epsilon_c$.

If $\omega(t)$ has nonanalyticicities at discerte points, then the WKB calculation must be done on the corresponding subintervals and then one has to multiply the corresponding transition matrices.

In other words, if $\omega(t)$ is analytic everywhere, the $\mu^2(\epsilon)$ is exponential everywhere, and in all other cases it is a power law.

5. Periodic $\omega(t)$

If $\omega(t)$ is periodic with period τ but otherwise completely general we can state some general rigorous results.

ω_0 at time t_0 and ω_1 at time $t_1 = t_0 + \tau$ are equal

Because $\mu^2/E_0^2 = \frac{1}{2}\left[\left(\frac{\bar{E}_1}{E_0}\right)^2 - 1\right]$

we see that $\bar{E}_1$ is always greater than E_0, that is, in a period τ, or any integer multiple of it, $T = n\tau$, the mean energy $\bar{E}_1$ never decreases.

If we denote by Φ_1 the transition map for one period, then the transition map Φ_n for an interval of exactly n periods of length τ is simply a power of Φ_1,

$\Phi_n = \Phi_1^n$.

If we use units such that $\omega_0 = \omega_1 = 1$ and $M = 1$, then elegantly

$\bar{E}_1 = \frac{E_0}{2}(\alpha + \beta) = \frac{E_0}{2}(a^2 + b^2 + c^2 + d^2) = \frac{E_0}{2}\mathrm{Tr}(\Phi\Phi^T)$

Let us decompose: $\Phi_1 = WDW^{-1}$ and $S = \mathrm{Tr}\Phi_1$

W is the transformation matrix and D is the diagonal matrix (e_1, e_2):

$e^2 - eS + 1 = 0$ and $e_1 = 1/e_2 = \frac{S}{2} \pm \sqrt{\left(\frac{S}{2}\right)^2 - 1}$.

Then we have: $\Phi = \Phi_n = \Phi_1^n = WD^nW^{-1}$.

$\bar{E}_1 \approx KE_0e_1^{2n}$ and $\mu^2 = \overline{(\Delta E_1)^2} \approx \frac{1}{2}\bar{E}_1{}^2 \approx \frac{K^2}{2}E_0^2e_1^{4n}$.

The contour $\mathcal{K}_0$ is topologically always a circle, it evolves into the closed curve $\mathcal{K}_n$ after the n-th full period, with the preserved, constant, area enclosed by $\mathcal{K}_n$.

If $|S| < 2$, $\mathcal{K}_n$ is rotating and oscillating with n.

If $|S| > 2$, $\mathcal{K}_n$ is exponentially stretched in direction $e_1 > 1$ and contracted in direction $e_2 < 1$ with n.

The energy of the individual initial condition will be exponentially increasing for *any* initial condition, except for the case when (q_0, p_0) is exactly in the direction e_2.

6. General formula for the energy evolution

We consider an exact expression for the evolution of the energy distribution by studying a decomposition of one adiabatic process into several consecutive adiabatic processes.

The energy distribution $P(E_1)$ evolved from the original delta-like distribution $\delta(E - E_0)$ is a kind of a Green function for the energy evolution. Let us denote it by $G(E_1; E_0)$.

If we have a spread of initial energies $w(E_0)$, the final energy distribution is $P(E_1) = \int G(E_1; E_0) w(E_0) dE_0$.

Thus by knowing G, which we call G-function, we can calculate the final energies of any family $w(E_0)$ of uniform canonical ensembles of initial conditions.

If the adiabatic process is ideal adiabatic, then the G-function is a delta function, $G(E_1; E_0) = \delta(E_1 - \omega_1 E_0 / \omega_0)$.

For ensembles of other types, which are not uniform canonical, we must go back to our fundamental equation and perform the averaging using the distribution in space (E_0, ϕ).

Now suppose that the interval of length T is divided into an arbitrary number of finite subintervals (t_j, t_{j+1}), where t_0 is the beginning of the process (interval) and t_n is the end of the process, and $j = 0, 1, \ldots, n-1$.

The behaviour of $\omega(t)$ inside each j-th subinterval is so far assumed to be entirely arbitrary, but the process must be such that at each integration step t_j the distribution is uniform canonical. This condition is certainly satisfied if the process is ideal adiabatic, in general not.

It is then obvious that the energy G-function $G(E; E_0)$ for the complete process divided into n subintervals is given by the multiple integral

$$G(E;E_0) = \underbrace{\int \ldots \int}_{n-1} G_n(E; x_{n-1}) G_{n-1}(x_{n-1}; x_{n-2}) \ldots G_1(x_1; E_0) dx_{n-1} \ldots dx_2 dx_1$$

All moments of the final distribution can be easily calculated as they are all fully determined by the first moment alone.

The first moment of any $G(E; E_0)$ is a linear function of the initial value E_0, namely

$\bar{E} = \int EG(E; E_0)dE = gE_0$

where the constant $g = (\alpha + \beta)/2$ is a constant independent of E_0 and is determined by the nature of $\omega(t)$ inside the relevant interval of evolution. We shall call g the g-factor of G.

We see: $g = g_n g_{n-1} \ldots g_2 g_1, \quad \bar{E} = gE_0 = g_n \ldots g_2 g_1 E_0$.

Obviously, for an ideal adiabatic process where each $g_j = \omega_j/\omega_{j-1}$, the above equation is certainly satisfied.

It is possible also to show the converse [31]: If the composition formula is true for *any* intermediate points of integration t_j and x_j, then the process must be ideal adiabatic, implying that

$$G(E_j; E_{j-1}) = \delta(E_j - \omega_j E_{j-1}/\omega_{j-1})$$

applies for all j, and $g_j = \omega_j/\omega_{j-1}$. This can be shown by splitting the time interval (t_0, t_n) into infinitesimal subintervals and using a piecewise constant function to approximate $\omega(t)$, and then using $g_j = \frac{1}{2}(\omega_j^2/\omega_{j-1}^2 + 1)$ from equation jump equation for all j, finally evaluating g by the previous factorization formula, and finding $g = \omega_n/\omega_0$, which implies that the process is ideal adiabatic at all times of the time interval, because $\mu^2 = 0$.

The composition formula (factorization property of the G-function) will apply also in nonlinear systems, but the relationship between $\bar{E}_1$ and E_0 is then no longer linear. Therefore using the composition formula for infinitesimal intervals, and approximating $\omega(t)$ by piecewise constant or piecewise linear functions etc. might be of extreme importance to find new global powerful approximations for G-functions and their moments.

The theory for nonlinear systems is left open for the future work.

7. Discussion and conclusions

• We have studied the time evolution of the energy in a general time-dependent 1D harmonic oscillator in a rigorous way, and then also calculated the final energy distribution $P(E_1)$ for a uniform canonical ensemble of initial conditions at energy E_0.

• $P(E_1)$ is universal and does not depend on the details of $\omega(t)$:

$P(E_1) = \frac{1}{\pi\sqrt{2\mu^2 - x^2}}$, where $x = E_1 - \bar{E}_1$

• We have calculated all moments of $P(E_1)$: Odd moments are exactly zero, the even moments are powers of the variance μ^2, which in turn is a function of the first moment $\bar{E}_1$. Therefore everything is determined by the first moment $\bar{E}_1$.

• The analysis clearly shows when the adiabatic invariant $I(t) = E(t)/\omega(t)$ is conserved or not. In the adiabatic limit $T \to \infty$ it is conserved. If it is not conserved, when T is finite, we calculate $\mu^2 \neq 0$ using WKB method analytically in closed form.

• We have also studied three specific solvable models and shown that the leading WKB term well describes the behaviour of μ^2 when $\epsilon = 1/T$ goes to zero.

• We have also shown what happens if $\omega(t)$ is smooth and of class $\mathcal{C}^m$, having m continuous derivatives: μ^2 oscillates as ϵ goes to zero, but in the mean vanishes as $\propto \epsilon^{2(m+1)}$.

• If $\omega(t)$ is analytic, thus it also is of class $\mathcal{C}^\infty$, it is known from the literature that μ^2 must decay exponentially $\propto \exp(const/\epsilon)$.

• If $\omega(t)$ is periodic, $\bar{E}_1$ can grow exponentially, and so does the variance μ^2, in which case $I(t) = E(t)/\omega(t)$ is not conserved, but we can describe the system.

• We have introduced the so-called G-function, which is a kind of a Green function for the evolution of the energy and derived a composition formula for it when the interval of evolution is decomposed into a finite number of subintervals, for which the corresponding G_j-function is known for all subintervals j.

This formula applies also to nonlinear systems and might be a good starting basis to describe them. The theory for nonlinear systems remains open and is a subject of the current research (Robnik and Romanovski 2000, [10]).

Knowing the G-function we can calculate $P(E_1)$ also for other families of initial uniform canonical ensembles with energy spread $w(E_0)$.

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