Limit cycles for 3-monomial differential equations

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Maribor, April, 2015

The problem

It is well known that

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R},$$

with P and Q polynomials can also be written as

$$\frac{dz}{dt} = \dot{z} = F(z, \bar{z}), \quad z \in \mathbb{C}, \quad t \in \mathbb{R},$$

where F is a complex polynomial.
We consider systems with F having few monomials and study the number of limit cycles of them.

The problem-II

Clearly, equations with one monomial

$$\dot{z} = Az^u\bar{z}^v$$

have NO limit cycles because they are homogeneous.
We are now studying equations with two monomials,

$$\dot{z} = Az^u\bar{z}^v + Bz^k\bar{z}^l,$$

where $A, B \in \mathbb{C}$ and $u, v, k, l \in \mathbb{N} \cup \{0\}$, trying to give a uniform bound for their number of limit cycles.
For instance, consider

$$\dot{z} = (1+i)\,z - z^2\bar{z}.$$

This equation with two monomials has the circle $|z| = 1$ as limit cycle, because, in polar coordinates, writes as $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$.

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First result

The aim of this talk is to present our results on equations with three monomials,

$$\dot{z} = Az^u\bar{z}^v + Bz^k\bar{z}^l + Cz^m\bar{z}^n,$$

where $A, B, C \in \mathbb{C}$ and $u, v, k, l, m, n \in \mathbb{N} \cup \{0\}$.

Theorem

For any $p \in \mathbb{N}$ there is a differential equation of type

$$\dot{z} = Az + Bz^k\bar{z}^l + Cz^m\bar{z}^n,$$

where $A, B, C \in \mathbb{C}$ and $k, l, m, n \in \mathbb{N} \cup \{0\}$, having at least p limit cycles.

Second result

Having previous result on mind it has sense to study the number of limit cycles for some fixed values k, l, m, n. we prove:

Theorem

Consider equation

$$\dot{z} = A\,z + B\,\bar{z} + C\,z^m\bar{z}^n. \tag{1}$$

Then for $m = 0$ *it has no limit cycles. For* $m \geq 2$, $\mathrm{Re}(A) \neq 0$ *and*

$$|B| \leq \frac{(m-1)|\,\mathrm{Re}(A)|}{m}, \tag{2}$$

it has at most one limit cycle. Moreover if the limit cycle exists it is hyperbolic and stable (resp. unstable) if $\mathrm{sgn}(\mathrm{Re}(A)) > 0$ *(resp.* < 0*) and it must surround the origin.*
Moreover, when (2) *does not hold there are equations of type* (1) *having more than one limit cycle*

The results of this talk are going to appear in:

A. Gasull, C. Li & J. Torregrosa. "Limit cycles for 3-monomial differential equations". *To appear in JMAA* (2015).

A well-known example with 3 monomials

A celebrated family of differential equations with 3 monomials is

$$\dot{z} = Az + Bz^2\bar{z} + C\bar{z}^{q-1},$$

with $q \geq 3$. It gives the *versal deformation* of a principal singular smooth systems having rotational invariance of $2\pi/q$ radians. The cases $q = 3, 4$ are called *strong resonances* while the cases $q \geq 5$ are called *weak resonances*. The situation $q \neq 4$ is well understood, see for instance:

- ARNOLD, V., "Chapitres supplémentaires de la théorie des équations différentielles ordinaires", Ed. Mir-Moscou, 1980.
- CHOW, S. N.; LI, C.; WANG, D., *A simple proof of the uniqueness of periodic orbits in the 1:3 resonance problem.* Proc. Amer. Math. Soc. **105** (1989) 1025–1032.
- HOROZOV, E., *Versal deformations of equivariant vector fields for the case of symmetries of order 2 and 3*. Trudy Sem. Pet., **5** (1979) 163–192 (in russian).

A well-known example with 3 monomials-II

The study of the limit cycles for case $q = 4$ turns out to be more difficult. To know the number of limit cycles surrounding the origin, and eventually surrounding also the other 4 or 8 critical point that the equation can posses is yet an open question. It is known that at least two limit cycles can exist surrounding the 9 critical points.

The problem of the number of limit cycles not surrounding the origin is solved by Zegeling. There are either NO limit cycles or exactly 4 hyperbolic ones, each one of them surrounding exactly one of the critical points of index $+1$.

Inspired by the presence of the four limit cycles not surrounding the origin for

$$\dot{z} = Az + Bz^2\bar{z} + C\bar{z}^{q-1},$$

and $q = 4$, we will consider a variation of the above system that will allow us to prove that there are equations with 3 monomials with an arbitrary large number of limit cycles non surrounding the origin.

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Some more references

- Álvarez, M. J.; Gasull, A.; Prohens R., *Limit cycles for cubic systems with a symmetry of order 4 and without infinite critical points.* Proc. Amer. Math. Soc., **136** (2008) 1035–1043.
- Guckenheimer, J., *Phase portraits of planar vector fields: computer proofs.* Exp. Mathematics, **4** (1995) 153–165.
- Krauskopf, B., *Bifurcation sequences at 1:4 resonance: an inventory.* Nonlinearity, **7** (1994) 1073–1091.
- Yu, P.; Han, M.; Yuan, Y., *Analysis on limit cycles of Z_q-equivariant polynomial vector fields with degree 3 or 4.* J. Math. Anal. Appl., **322** (2006) 51–65.
- Zegeling, A., *Equivariant unfoldings in the case of symmetry of order 4.* Serdica, **19** (1993) 71–79.

Starting equation for proving our first theorem

Lemma

Consider the polynomial differential equation

$$\dot{z} = F(z, \bar{z}) + Cz^m\bar{z}^n,$$

where $0 \neq C \in \mathbb{C}$ *and* $\deg(F) < m + n$. *Then, the sum of the indices of all its critical points is* $m - n$.

The condition on differential equation

$$\dot{z} = Az + Bz^2\bar{z} + C\bar{z}^{q-1}, \qquad (3)$$

to be Hamiltonian is $\mathrm{Re}(A) = \mathrm{Re}(B) = 0$. By the Lemma, for $q \geq 5$ the total sum of the indices of all its critical points in this case is $1 - q < 0$. Moreover, it can be seen that it has only one critical point of index $+1$. Therefore, for $q \geq 5$, the Hamiltonian systems in (3) are not good candidates to start our study.

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Starting equation for proving our first theorem-II

Instead of considering differential equation of the form

$$\dot{z} = Az + Bz^2\bar{z} + C\bar{z}^{q-1},$$

we take the following subclass of equations with 3 monomials,

$$\dot{z} = A\,z + B\,z^{p-1}\bar{z}^{p-2} + C\,\bar{z}^{p-1} = A\,z + B\,z|z|^{2(p-2)} + C\,\bar{z}^{p-1}, \quad (4)$$

with $p \geq 3$, which also have rotational invariance of $2\pi/p$ radians. Notice that both coincide when $p = q = 3$.

Starting equation for proving our first theorem-III

Equation

$$\dot{z} = (a+i)\,z + (b+i)\,z^{p-1}\bar{z}^{p-2} - \frac{5i}{2}\,\bar{z}^{p-1},$$

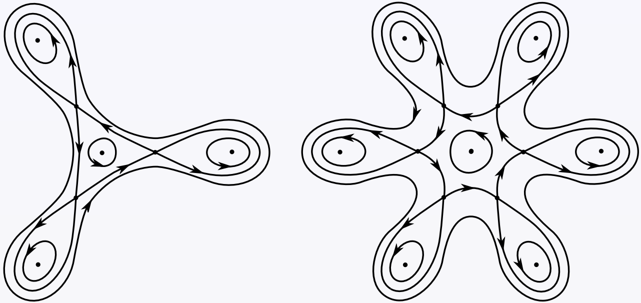
when $a = b = 0$ is Hamiltonian, with Hamiltonian function

$$H(r,\theta) = \frac{r^2}{2} - \frac{5}{2p} r^p \cos(p\,\theta) + \frac{r^{2(p-1)}}{2(p-1)} - \tilde{\rho},$$

where $\tilde{\rho} = \frac{(p-2)(p-5)}{2p(p-1)}\,2^{\frac{2}{p-2}}$.

Starting equation for proving our first theorem-IV

Their phase portraits are:



Centers when $a = b = 0$ for the cases $p = 3$ and $p = 6$.

Proof of the first theorem

Our first theorem is a corollary of the following proposition:

Proposition

For $3 \leq p \in \mathbb{N}$, consider the 2-parameter family of systems

$$\dot{z} = (a+i)\,z + (b+i)\,z|z|^{2(p-2)} - \frac{5i}{2}\,\bar{z}^{p-1},$$

with $a, b \in \mathbb{R}$, $3 \leq p \in \mathbb{N}$. Then there exist values for a and b for which the above equation has at least p limit cycles.

Proof of the proposition

The differential equation in polar coordinates is

$$dH(r,\theta) - (a\,r^2 + b\,r^{2(p-1)})\,d\theta = 0.$$

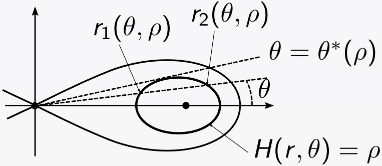
Writing $a = \varepsilon\,\alpha$ and $b = \varepsilon\,\beta$, for $\alpha, \beta \in \mathbb{R}$ and ε small enough, the associated first order Melnikov function is

$$M(\rho) = \alpha\, I_2(\rho) + \beta\, I_{2(p-1)}(\rho),$$

where

$$I_j(\rho) = \int_{H=\rho} r^j\, d\theta = 2\int_0^{\theta^*(\rho)} \left(r_2^j(\theta,\rho) - r_1^j(\theta,\rho) \right)\, d\theta,$$

for $j = 2, 2(p-1)$ and $\rho \in (\rho^*, 0)$.



Then, we introduce the auxiliary analytic function

$$J(\rho) = \frac{I_{2(p-1)}(\rho)}{I_2(\rho)}, \quad \rho \in (\rho^*, 0)$$

and we write

$$M(\rho) = I_2(\rho)\big(\alpha + \beta J(\rho)\big).$$

Notice that $I_2(\rho) > 0$ because this function gives the double of the area surrounded by a connected component of the curve $H(r, \theta) = \rho$.

We claim that $J(\rho)$ is not constant. Let us prove first that if the claim holds then the proposition is already proved.

Proof of the proposition-III

Recall that

$$M(\rho) = I_2(\rho)\big(\alpha + \beta J(\rho)\big).$$

If J is not a constant, take $\widehat{\rho} \in (\rho^*, 0)$. Then choosing $\alpha = -J(\widehat{\rho})$ and $\beta = 1$ we have that $M(\widehat{\rho}) = 0$. Since J is not a constant and the function is analytic, this zero of M has a given finite multiplicity. If this zero is simple we are done.

If not, by using again that $I_2(\rho) > 0$ it is easy to see that taking ν small enough, and with the suitable sign, the function

$$M(\rho) = (\nu - J(\widehat{\rho}))\, I_2(\rho) + I_{2p-2}(\rho)$$

has a simple zero $\widehat{\rho}_\nu$, near $\rho = \widehat{\rho}$, as we wanted to prove.

Recall that

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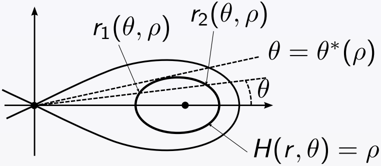
Proof of the claim

We want to prove that

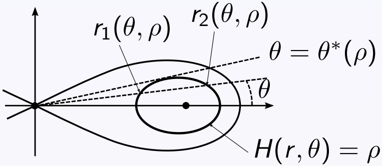
$$J(\rho) = \frac{I_{2(p-1)}(\rho)}{I_2(\rho)}, \quad \rho \in (\rho^*, 0)$$

is not constant.
If J was constant, we start computing its value.
We first parameterize the oval $H(r, \theta) = \rho$ in polar coordinates, see the figure.



For each $\theta \in [-\theta^*(\rho), \theta^*(\rho)]$ the values of r are $r_1(\theta, \rho)$ and $r_2(\theta, \rho)$, with $r_1(\theta, \rho) \leq r_2(\theta, \rho)$. The equality between both r_j only holds for $\theta = \pm\theta^*(\rho)$.

Proof of the claim-II



By the mean value theorem for integrals,

$$I_2(\rho)=\int_{H=\rho} r^2\,d\theta = 2\int_0^{\theta^*(\rho)} \left(r_2^2(\theta,\rho)-r_1^2(\theta,\rho)\right)\,d\theta$$

$$= 2\left(r_2(\overline{\theta}(\rho),\rho)+r_1(\overline{\theta}(\rho),\rho)\right)\int_0^{\theta^*(\rho)} \left(r_2(\theta,\rho)-r_1(\theta,\rho)\right)\,d\theta,$$

for some $\overline{\theta}(\rho)\in(0,\theta^*(\rho))$.

Similarly,

$$I_{2p-2}(\rho) = \int_{H=\rho} r^{2p-2}\, d\theta = 2\int_0^{\theta^*(\rho)} \left(r_2^{2p-2}(\theta,\rho) - r_1^{2p-2}(\theta,\rho) \right) d\theta$$

$$= 2\sum_{j=0}^{2p-3} r_2^{2p-3-j}(\widehat{\theta}(\rho),\rho)\, r_1^{j}(\widehat{\theta}(\rho),\rho) \int_0^{\theta^*(\rho)} \left(r_2(\theta,\rho) - r_1(\theta,\rho) \right) d\theta,$$

for some $\widehat{\theta}(\rho) \in (0, \theta^*(\rho))$. Moreover,

$$\lim_{\rho\to 0} \overline{\theta}(\rho) = \lim_{\rho\to 0} \widehat{\theta}(\rho) = 0,$$

and

$$\lim_{\rho\to 0} r_j(\overline{\theta}(\rho),\rho) = \lim_{\rho\to 0} r_j(\widehat{\theta}(\rho),\rho) = 2^{1/(p-2)}, \quad j = 1, 2.$$

Proof of the claim-IV

Hence,

$$\lim_{\rho\to 0} J(\rho) = \lim_{\rho\to 0} \frac{\sum_{j=0}^{2p-3} r_2^{2p-3-j}(\widehat{\theta}(\rho),\rho)\, r_1^{j}(\widehat{\theta}(\rho),\rho)}{r_2(\overline{\theta}(\rho),\rho)+r_1(\overline{\theta}(\rho),\rho)}$$

$$= \frac{(2p-2)\left(2^{1/(p-2)}\right)^{2p-3}}{2\cdot 2^{1/(p-2)}} = 4(p-1).$$

So, we will assume to arrive to a contradiction, that $J(\rho) \equiv 4(p-1)$.

Proof of the claim-V

If $J(\rho) \equiv 4(p-1)$, then

$$J'(\rho) = \frac{I'_{2p-2}(\rho) I_2(\rho) - I_{2p-2}(\rho) I'_2(\rho)}{I_2^2(\rho)} \equiv 0.$$

Then, it also holds that

$$\frac{I'_{2p-2}(\rho)}{I'_2(\rho)} \equiv 4(p-1). \tag{5}$$

By the Gelfand–Leray formula,

$$I'_{2p-2}(\rho) = \int_{H=\rho} (2p-2) r^{2p-3} \frac{\partial r}{\partial \rho} \, d\theta.$$

Proof of the claim-VI

We know that

$$I'_{2p-2}(\rho) = \int_{H=\rho} (2p-2) r^{2p-3} \frac{\partial r}{\partial \rho} \, d\theta.$$

Since $H(r(\theta,\rho),\theta)) = \rho$ for all ρ, we get that

$$\frac{\partial H(r(\theta,\rho),\theta)}{\partial r} \frac{\partial r(\theta,\rho)}{\partial \rho} = 1.$$

Hence, using the differential equation in polar coordinates,

$$\frac{\partial r(\theta,\rho)}{\partial \rho} = \left(\frac{\partial H(r(\theta,\rho),\theta)}{\partial r} \right)^{-1} = \frac{1}{r(\theta,\rho)} \frac{dt}{d\theta}.$$

Therefore, we can parameterize the Abelian integrals using the variable t.

Proof of the claim-VII

Therefore, from

$$I'_{2p-2}(\rho) = \int_{H=\rho} (2p-2) r^{2p-3} \frac{\partial r}{\partial \rho}\, d\theta.$$

and

$$\frac{\partial r(\theta,\rho)}{\partial \rho} = \frac{1}{r(\theta,\rho)} \frac{dt}{d\theta}.$$

we get that

$$I'_{2p-2}(\rho) = (2p-2) \int_0^{T(\rho)} r^{2p-4}(t)\, dt,$$

where $r(t)$ denotes the time parametrization of the periodic orbit contained in $H(r,\theta) = \rho$ (for shortness, we omit the dependence with respect to ρ) and $T(\rho)$ is its period. Similarly,

$$I'_2(\rho) = 2 \int_0^{T(\rho)} dt = 2\, T(\rho).$$

Proof of the claim-VIII

As a consequence of the expressions of $I'_{2p-2}(\rho)$ and $I'_2(\rho)$ we get that

$$J(\rho) \equiv 4(p-1) \Leftrightarrow \frac{I'_{2p-2}(\rho)}{I'_2(\rho)} \equiv 4(p-1)$$

and also is equivalent to

$$\frac{\displaystyle\int_0^{T(\rho)} r^{2p-4}(t)\,dt}{T(\rho)} = 4,$$

that can be written as

$$\int_0^{T(\rho)} G_\rho(t)\,dt = 0, \quad \text{where} \quad G_\rho(t) := r^{2p-4}(t) - \left(2^{1/(p-2)}\right)^{2p-4},$$

or by symmetry, as

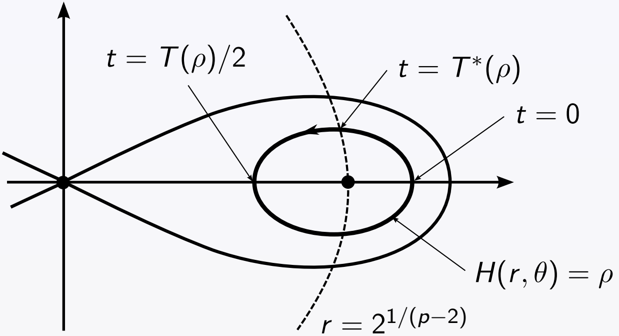
$$K(\rho) := \int_0^{T(\rho)/2} G_\rho(t)\,dt = 0,$$

Proof of the claim-IX

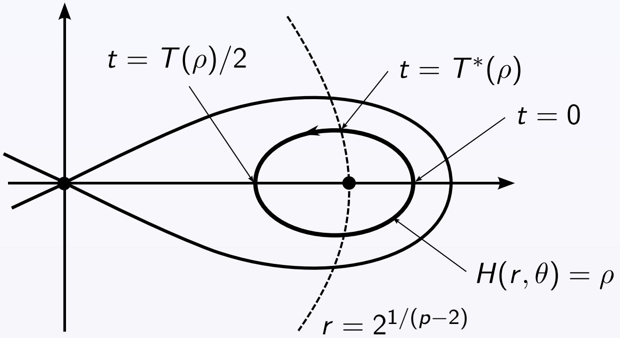
We have that

$$K(\rho) := \int_0^{T(\rho)/2} G_\rho(t)\,dt = 0, \qquad (6)$$

where $G_\rho(t) := r^{2p-4}(t) - \left(2^{1/(p-2)}\right)^{2p-4}$.



We will prove that equality (6) is false for ρ near ρ^* (energy of the loop).



We introduce the time $t = T^*(\rho)$. Then, $K(\rho) = K^+(\rho) + K^-(\rho)$, where

$$K^-(\rho) = \int_{T^*(\rho)}^{T(\rho)/2} G_\rho(t)\, dt > 0, \quad K^+(\rho) = \int_0^{T^*(\rho)} G_\rho(t)\, dt < 0.$$

Moreover, the function $K^+(\rho)$ has an upper bound and $\lim_{\rho \to \rho^*} K^-(\rho) = -\infty$, which provides a contradiction.

Proof of the second theorem

Theorem

$$\dot{z} = A\,z + B\,\bar{z} + C\,z^m \bar{z}^n.$$

Then for $m = 0$ *it has no limit cycles. For* $m \geq 2$, $\operatorname{Re}(A) \neq 0$ *and*

$$|B| \leq \frac{(m-1)|\operatorname{Re}(A)|}{m},$$

it has at most one limit cycle. Moreover if the limit cycle exists it is hyperbolic and stable (resp. unstable) if $\operatorname{sgn}(\operatorname{Re}(A)) > 0$ *(resp.* < 0*) and it must surround the origin.*
Moreover, when (2) *does not hold there are equations of type* (1) *having more than one limit cycle*

Our proof is based on showing that all possible limit cycles, all points of index +1 (except the origin), all nodal sectors of semi-hyperbolic critical points and all polycycles have the same stability.

Proof of the second theorem-II

We fix $\mathrm{Re}(A) > 0$. The case $\mathrm{Re}(A) < 0$ can be studied similarly.
Assume that we have already proved the following facts:

(a) Focus and node points different from the origin are attractor.

(b) The origin is an unstable focus or node.

(c) Saddle-node points have the nodal sector of attracting type.

(d) All periodic orbits are hyperbolic and attractive limit cycles.

(e) All polycycles are attractors.

Then it is not difficult to prove the uniqueness of the limit cycle.

Proof of the second theorem-III

Consider a periodic orbit γ of the differential equation and denote by $\mathcal{D}$ the bounded region that it surrounds. First, we prove that the origin must be in $\mathcal{D}$.

Assume, to arrive to a contradiction, that the origin is not in $\mathcal{D}$. Then there are only a measure zero set of points in $\mathcal{D}$ that can have α-limit (the unstable manifolds of the saddle and saddle-node points). This result is in contradiction with the Poincaré–Bendixson theory. Hence all periodic orbits must surround the origin, and eventually other critical points.

To end the proof let us show that there is at most one limit cycle surrounding the origin. Assume that there were two, γ_1 and γ_2. Then arguing as in the previous case but on the annular bounded region with boundary $\gamma_1 \cup \gamma_2$ we arrive again to a contradiction.

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Proof of the second theorem-IV

Let us prove first that all periodic orbits are hyperbolic and attractive.

It is well-known that the hyperbolicity and the stability of a given limit cycle, $z(t) = x(t) + iy(t)$, of period T of a vector field X is controlled by its characteristic exponent,

$$\sigma = \int_0^T \operatorname{div}(X)(x(t), y(t))\, dt.$$

Writing it in polar coordinates,

$$\sigma = \int_0^T \left(\frac{1}{r} \frac{\partial(rR(r,\theta))}{\partial r} + \frac{\partial \Phi(r,\theta)}{\partial \theta} \right) (r(t), \theta(t))\, dt.$$

The expression in polar coordinates of our equation is

$$\dot r = R(r,\theta) = \operatorname{Re}(A + S(\theta))\, r + \operatorname{Re}(U(\theta))\, r^{m+n},$$
$$\dot\theta = \Phi(r,\theta) = \operatorname{Im}(A + S(\theta)) + \operatorname{Im}(U(\theta))\, r^{m+n-1},$$

where

$$S(\theta) = B\, \mathrm{e}^{-2i\theta}, \quad U(\theta) = C\, \mathrm{e}^{(m-n-1)i\theta}.$$

Proof of the second theorem-V

Afer some computations

$$\sigma = \int_0^T 2\,\mathrm{Re}(A) + 2m\,\mathrm{Re}\left(U(\theta(t))\right) r^{m+n-1}(t)\,dt. \qquad (7)$$

Using that $r = r(t), \theta = \theta(t)$ is a T-periodic orbit we get that

$$0 = \int_0^T \frac{\dot r(t)}{r(t)}\,dt = \int_0^T \mathrm{Re}\left(A + S(\theta(t))\right) + \mathrm{Re}\left(U(\theta(t))\right) r^{m+n-1}(t)\,dt,$$

and we can write (7) as

$$\sigma = \int_0^T 2(1-m)\,\mathrm{Re}(A) - 2m\,\mathrm{Re}\left(S(\theta(t))\right) dt$$
$$= 2m \int_0^T \frac{1-m}{m}\,\mathrm{Re}(A) - \mathrm{Re}\left(B\,\mathrm{e}^{-2i\theta(t)}\right) dt.$$

Clearly, since $(1-m)\,\mathrm{Re}(A) \neq 0$, if $|B| \leq (m-1)|\,\mathrm{Re}(A)|/m$, the above integrand does not change sign, and hence $\sigma \neq 0$ and $\mathrm{sgn}(\sigma) = -\,\mathrm{sgn}(\mathrm{Re}(A))$, as we wanted to prove.

Proof of the second theorem-VI

The stability of the critical points can be studied in a similar way. We get that

$$\operatorname{div}\left(X(r^*,\theta^*)\right) = 2m\Big(\frac{1-m}{m}\operatorname{Re}(A) - \operatorname{Re}\left(B\,\mathrm{e}^{-2i\theta^*}\right)\Big).$$

Therefore, under the hypotheses of the statement,

$$\operatorname{sgn}\left(\operatorname{div}\left(X(r^*,\theta^*)\right)\right) = -\operatorname{sgn}(\operatorname{Re}(A)).$$

Moreover, the determinant of the differential of X is $\det((\mathrm{d}\,X)_{(0,0)}) = |A|^2 - |B|^2$ which is positive because

$$|A|^2 - |B|^2 > |A|^2 - \left(\frac{m}{m-1}\right)^2 |B|^2 \geq |A|^2 - \left(\operatorname{Re}(A)\right)^2 \geq 0.$$

Finally we study the stability of the polycycles. Their corners are formed by hyperbolic saddles or semi-hyperbolic saddles or saddle-nodes. Since the divergence at them has always the sign of $-\operatorname{Re}(A)$, we can prove that all the polycycles are stable (resp. unstable) when $\operatorname{Re}(A) > 0$ (resp. $\operatorname{Re}(A) < 0$).

This ends the proof of the theorem

We have used the following result:

Proposition

Let Γ be a polycycle of an analytic vector field X with elementary corners $u_1, u_2, \ldots, u_\ell$ and such that $\operatorname{div}(X(u_j)) < 0$ (resp. > 0) for all j. Then Γ is an attracting (resp. repelling) polycycle.

Its proof is based on:

BAMÓN, R.; MARTIN-RIVAS, J C.; MOUSSU, R., *Sur le problème de Dulac. (French) [On Dulac's problem]* C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), 737–739.

Proof of the second theorem-VII

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Stability of the polycicles. Idea of the proof

It is well-known, that when all the critical points at the corners are hyperbolic, with eigenvalues $-\lambda_j < 0 < \mu_j$, then Γ is stable (respectively, unstable) if $\rho(\Gamma) < 1$ (respectively, $\rho(\Gamma) > 1$), where

$$\rho(\Gamma) = \prod_{j=1}^{\ell} \frac{\mu_j}{\lambda_j}.$$

Fix for instance the case where, for $j = 1, \dots, \ell$, $\mathrm{div}(X(u_j)) < 0$. Then, for all j, $\mu_j/\lambda_j < 1$ and the proposition follows.

In general, when either $\rho(\Gamma) = 1$ or there are semi-hyperbolic corners, the stability of the corresponding polycycles can be very hard to determine. In particular, for each semi-hyperbolic corner u, its associated local return Dulac map is flat (resp. vertical) when $\mathrm{div}(X(u)) < 0$ (resp. > 0). Recall that flat return maps have all their derivatives zero at the origin and that vertical maps are the inverse of flat maps. The really difficult situation appears when flat and vertical local return maps coexist. Fortunately, under our hypotheses all these maps are of the some type.

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Examples with limit cycles

Lemma

Consider differential equation

$$\dot{z} = \big(\varepsilon\,(1-\lambda) + i\big)\, z - \varepsilon\,(1+\lambda)\,\bar{z} - \frac{1}{2}\, i\, z^2, \tag{8}$$

with $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ a small parameter. Then the following holds:

(i) *When $\lambda \in (-1/3, 0)$ and ε is small enough, it has a limit cycle surrounding the critical point that when $\varepsilon = 0$ is at $z = 2$.*

(ii) *When $\lambda \in [-3, -1/3]$, it is under the hypotheses of our Theorem. Hence it has no limit cycles not surrounding the origin.*

Examples with limit cycles-II

Lemma

There are equations of the form

$$\dot{z} = A\,z + B\,\bar{z} + z^3, \tag{9}$$

having at least two limit cycles, each one of them surrounding a different critical point.

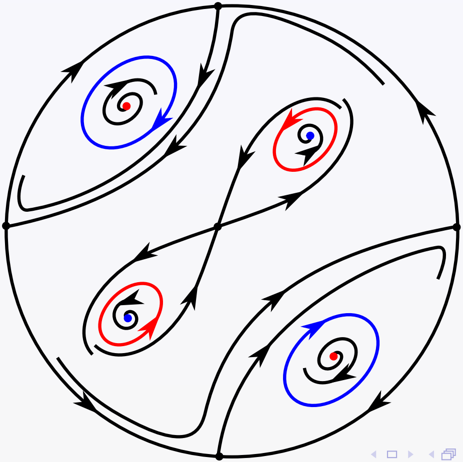
We consider the 1-parameter family,

$$\dot{z} = (-9+28\,\varepsilon^2+16\,\varepsilon^3+(4+8\,\varepsilon)i)\,z+(10+18\,\varepsilon+16\,\varepsilon^2+8\,\varepsilon^3)\,\bar{z}+(4+8\,\varepsilon)\,z^3, \tag{10}$$

and prove that for $\varepsilon > 0$, small enough, a hyperbolic attracting limit cycle is born via an Andronov–Hopf bifurcation and this limit cycle surrounds the point $z_\varepsilon^+ = 1 + (1/2 + \varepsilon)i$. Since the equation is invariant by the change of variables $z \to -z$, a second symmetric limit cycle appears surrounding another critical point.

Examples with limit cycles-III

Finally, our numerical simulations also show that differential equation $\dot{z} = A\,z + B\,\bar{z} + z^3$,, with $A = 1 + 9i/4$, $B = 13/4 + i/2$ has at least four limit cycles, see its phase portrait on the Poincaré disc.



Thank you very much
for your attention!