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Theory of Adiabatic Invariants
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Abstract. We review the basic theory on adiabatic invariants in classical and quantum mechanics.

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Extended abstract

Adiabatic invariants, denoted by I, are approximate constants (invariants) of motion of a given dynamical system (not necessarily Hamiltonian), which are approximately preserved (constant) during a process of very slow change of the system's parameters (denoted by λ), on a time scale T, which is supposed to be much larger than any typical dynamical time scale such as traversal time or the period of the shortest periodic orbits etc.

This is an asymptotic statement, in the sense that the adiabatic invariants are the better preserved the slower the driving of the system, that is the more slowly the switching function $\lambda = \lambda(t)$ varies on the typical evolutionary time scale T, and the preservation is perfect in the limit $T \to \infty$.

The important point is that whilst the system's parameters $\lambda(t)$ and their dynamical quantities like the total energy, angular momentum etc, can change by arbitrarily large amounts, their combination involved in the adiabatic invariant I is very well preserved to a very high degree of accuracy, and that allows us to calculate changes of important quantities in dynamical systems. Examples are in celestial mechanics, in other Hamiltonian systems, in motion of charged particles in magnetic and electric fields, etc.

The accuracy of preservation can be calculated in one degree of freedom systems and is exponentially good with T if the switching function $\lambda(t)$ is analytical (of class $\mathcal{C}^\infty$), that is to say the change of the adiabatic invariant ΔI is of the form

$$\Delta I = \alpha \exp(-\beta T), \tag{1}$$

where α and β are certain constants. If, however, the switching function $\lambda(t)$ is only of class $\mathcal{C}^m$ (m-times continuously differentiable), then the change of the adiabatic invariant ΔI during an adiabatic change over a time period of length T is algebraic only, namely

$$\Delta I = \alpha T^{-(m+1)}. \tag{2}$$

In both cases $\Delta I \to 0$ as $T \to \infty$.

The fact that the evolutionary time scale T is very large compared to the typical shortest dynamical time scales (average return time etc), points to the intuitive idea of the averaging method or the so-called averaging principle. Here it is thought that the long-term evolution (adiabatic evolution) of the system can be calculated by replacing the actual dynamical system with its averaged correspondent, obtained by averaging it over the shortest dynamical time scales, i.e. over the fast variables. Such a procedure is very well known e.g. in celestial mechanics where the secular effects of the third-body perturbations of a planet are obtained by averaging the perturbations over one revolutionary period of the perturbers. This has been done already by Gauss in the context of studying the dynamics of planets.

The adiabatic invariants can be easily calculated in one dimensional systems, and in completely integrable systems with N degrees of freedom, and something is known about the ergodic Hamiltonian systems, whilst little is known about the adiabatic invariants in mixed type Hamiltonian systems (with divided phase space), where for some initial conditions in the classical phase space we have regular motion on invariant tori and irregular (chaotic) motion for other (complementary) initial conditions.

One elementary example is the simple (mathematical) pendulum, of point mass m and of length l with the declination angle φ, described by the Hamiltonian

$$H = \frac{p_\varphi^2}{2ml^2} - mgl\cos\varphi \tag{3}$$

where $p_\varphi = ml^2\dot{\varphi}$ is the angular momentum, and for small oscillations $\varphi \ll 1$, around the stable equilibrium $\varphi = 0$, it is described by the harmonic Hamiltonian

$$H' = \frac{p_\varphi^2}{2ml^2} + \frac{mgl}{2}\varphi^2. \tag{4}$$

Here the angular oscillation frequency is $\omega = 2\pi\nu = \sqrt{g/l}$, where ν is the frequency, and g is the gravitational acceleration. By E we denote the total energy of the Hamiltonian H'. It has been discovered by Ehrenfest, that the quantity $I = E/\omega$ is the adiabatic invariant of the system, so the change of $E(t)$ on adiabatic evolutionary very large time scale $T \gg 1/\nu$ is such that $I = E(t)/\omega(t)$ remains constant. Therefore, if for example the length of the pendulum $l = l(t)$ is slowly, adiabatically, changing, then the energy of the system will change according to the law

$$E = E_0\sqrt{\frac{l_0}{l}}, \tag{5}$$

where E_0 and l_0 are the initial and E and l the final values of the two variables. One can easily show that the oscillation amplitude φ_0 changes as $\propto l^{-3/4}$ as the length l

changes. This is an elementary example of a dynamically driven system, in which the change of energy E can be very large, as is the change of ω, but $I = E/\omega$ is a well preserved adiabatic invariant, in fact it is exponentially well preserved if the switching function $\omega(t)$ is analytic.

More generally, for Hamiltonian systems $H(q, p, \lambda)$, with one degree of freedom, whose state is described by the coordinate q and canonically conjugate momentum p in the phase space (q, p), and $\lambda = \lambda(t)$ is the system's parameter, in general slowly changing on time scale T, one can show that the action integral

$$I(E, \lambda) = I(E(t), \lambda(t)) = \frac{1}{2\pi} \oint p dq \tag{6}$$

is the adiabatic invariant of the system, where the contour integral is taken at fixed total energy E and fixed value of λ. In this case $2\pi I$ is interpreted as the area inside the curve $E = const.$ in the phase plane (q, p). The accuracy is exponentially good if $\lambda(t)$ is an analytic function and algebraic if it is of class C^m. And the theorem holds true only if the frequency ω is nonzero.

This implies that a passage through a separatrix (in the phase space of a one dimensional system) is excluded, because $\omega = 0$ there, and indeed in such a case of crossing a separatrix a different approach is necessary with highly nontrivial result. When crossing a separatrix of a one-dimensional double potential well from outside in an adiabatic way going inside, a bifurcation takes place, and the capture of the trajectory in either of the two wells is possible with some probabilities. These probabilities can be calculated quite easily, and also the spread of the adiabatic invariant ΔI after such a passage can be calculated, but this is more difficult. The important applications are in celestial mechanics, where an adiabatic capture of a small body near a resonance with a planet can take place, in plasma physics, in quantum mechanics of states close to the separatrix (in the semiclassical limit), etc.

This is an interesting result, because I is precisely that quantity which according to the "old quantum mechanics" of Bohr and Sommerfeld has to be "quantized", i.e. put equal to an integer multiple of Planck's constant $\hbar$. Of course, the "old quantum mechanics" is generally wrong, but it can be a good approximation to the solution of the Schrödinger equation. Even then, strictly speaking, the quantization condition in the sense of EBK or Maslov quantization, must be written in the form

$$I = \frac{1}{2\pi} \oint p dq = (n + \frac{\alpha}{4})\hbar \tag{7}$$

where $n = 0, 1, 2, \ldots$ is the quantum number and α is the Maslov index, i.e. number of caustics (projection singularities) round the cycle $E = const.$ in the phase plane. For smooth systems with quadratic kinetic energy it is typically $\alpha = 2$. Thus at this semiclassical level we have the semiclassical adiabatic invariant, saying that in one dimensional systems under an adiabatic change the quantum number (and thus the eigenstate) is preserved. This agrees with the exact result in the theory of the Schrödinger equation in quantum mechanics. Round a closed loop in a parameter space,

a quantum system returns to its original state, except for the phase! (This closed-loop phase change is essentially the so-called Berry's phase.)

The method of averaging can be used also in N dimensional Hamiltonians $H = H(\mathbf{q}, \mathbf{p})$, where $\mathbf{q}$ and $\mathbf{p}$ are N-dimensional vectors, but it works only in two extreme cases, the integrable case and the ergodic case.

In a classical integrable Hamiltonian system we have N analytic, global and functionally independent constants of motion $A_i = A_i(\mathbf{q}, \mathbf{p})$, $i = 1, 2, \ldots, N$, pairwise in involution, i.e. all Poisson brackets $\{A_i, A_j\}$ vanish identically everywhere in phase space. The orbits in phase space are then confined to an invariant N-dimensional surface, and according to the Liouville-Arnold theorem the topology of these surfaces must be the topology of an N dimensional torus. Then an action integral $I = \frac{1}{2\pi} \oint \mathbf{p}.d\mathbf{q}$ along a closed loop on a torus will be zero if the loop can be continuously shrunk to a point on the torus. But there are loops which cannot be shrunk to a point due to the topology of the torus. Then the integral I is different from zero, but otherwise its value does not depend on the particular loop, so in a sense it is a topological invariant of the torus. On an N dimensional torus there are N such independent elementary closed loops C_i, $i = 1, 2, \ldots, N$. The integrals which we call simply actions or action variables

$$I_i = \frac{1}{2\pi} \oint_{C_i} \mathbf{p}.d\mathbf{q} \tag{8}$$

are then the most natural momentum variables on the torus, whilst angle variables $\mathbf{\Theta}$ specifying the position on the torus labeled by $\mathbf{I}$ can be generated from the transformation

$$\mathbf{\Theta} = \frac{\partial S(\mathbf{I}, \mathbf{q})}{\partial \mathbf{I}}, \tag{9}$$

where $S = \int \mathbf{p}.d\mathbf{q}$ is an action integral on the torus.

Applying the averaging principle (the method of averaging) one readily shows that for an integrable system the actions $\mathbf{I}$ are N adiabatic invariants, provided the system is nondegenerate, which means that the frequencies

$$\omega = \frac{\partial H}{\partial \mathbf{I}} \tag{10}$$

on the given torus are not rationally connected, that is to say there is no such integer vector $\mathbf{k}$ that $\omega.\mathbf{k} = 0$. The problem is that during an adiabatic process the frequencies ω will change and therefore strictly speaking there will be infinitely many points of $\lambda = \lambda(t)$, where $\omega.\mathbf{k} = 0$, which will, strictly speaking, invalidate the theorem. However, it is thought that if the degree of resonances or rationality conditions $\omega.\mathbf{k} = 0$ is of very high order, meaning that all components of $\mathbf{k}$ are very large, that then the adiabatic invariants I_i will be quite well preserved. But low order resonances (rationality conditions) must be excluded. The details of such a process still call for further investigation.

When the N actions I_i of an integrable system are quantized in the sense of Maslov, as explained before in the one-dimensional case, we again find the agreement, at this

semiclassical level, with the quantum mechanics: In a family of integrable systems all N quantum numbers and the corresponding eigenstates are preserved under an adiabatic change.

Another extreme of classically ergodic and thus fully chaotic systems has been considered already by Hertz. He found that in such ergodic Hamiltonian systems the phase space volume enclosed by the energy surface $H(\mathbf{q},\mathbf{p}) = E =$ constant, is the adiabatic invariant, denoted by

$$\Omega(E) = \int_{H(\mathbf{q},\mathbf{p})\leq E} d^N\mathbf{q}d^N\mathbf{p}. \tag{11}$$

Of course, here it is required that while system's parameter $\lambda(t)$ is slowly changing the system itself must be ergodic for all $\lambda(t)$. Sometimes this condition is difficult to satisfy, but sometimes it is easily fulfilled. Examples are the stadium of Bunimovich with varying length of the straight line between the two semicircles, or Sinai billiard with varying radius of the circle inside a square. For an ergodic two dimensional billiard of area $\mathcal{A}$ and point mass m, we have

$$\Omega(E) = 2\pi m\mathcal{A}E. \tag{12}$$

Therefore when $\mathcal{A}$ is adiabatically changing, the energy E of the billiard particle is changing reciprocally with $\mathcal{A}$. Diminishing $\mathcal{A}$ implies increasing E, and this can be interpreted as work being done against the "pressure" of only one particle, if we define the pressure as the time average of the momentum transfer at collisions with the boundary of our ergodic billiard. In fact, there is a formalism to proceed with this analysis close to the thermodynamic formalism, as derived from the statistical mechanics, except that here we are talking about the time averages rather than phase averages of classical variables.

Again, this general result for ergodic systems is interesting from the quantal point of view, because $\mathcal{N} = \Omega(E)/(2\pi\hbar)^N$ is precisely the number of energy levels below the energy E, in the semiclassical limit of very large $\mathcal{N}$, which is well known as the Thomas-Fermi rule. It is the number of elementary quantal Planck cells inside the volume element $H(\mathbf{q},\mathbf{p}) \leq E$. Indeed, quantum mechanically, the eigenstate and the (energy counting sequential) main quantum number $\mathcal{N}$ is preserved under an adiabatic change.

In case of a mixed type Hamiltonian system, which is a typical case in nature, the adiabatic theory is in its infancy. Moreover, in three or higher degrees of freedom, we have the universal diffusion on Arnold web, which is dense on the energy surface, even for KAM type Hamiltonian systems which are very close to the integrability, like our solar planetary system. On the Arnold web we have a diffusional chaotic motion, and there is a rigorous theory by Nekhoroshev giving a rigorous upper bound to the diffusion rate in such case. However, when compared with the numerical calculations, it is found that the difussion rate is many orders of magnitude smaller than the Nekhoroshev bound. In other words, the actual diffusion time is much longer than estimated by Nekhoroshev,

implying that there we have some approximate adiabatic invariant for long times, but not too long times.

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