Econophysics V:
Credit Risk

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Outline

- Introduction
- Market risk versus Credit risk
- Reduced form models versus Structural models
- Loss distribution
- Numerical simulations and Random matrix approach
- Conclusions: general, present credit crisis
Introduction
Diversification in a stock portfolio, no correlations

- empirical distribution of normalized returns (400 stocks)
Diversification in a stock portfolio, no correlations

> empirical distribution of normalized returns (400 stocks)
> portfolio: superposition of stocks
Diversification in a stock portfolio, no correlations

- empirical distribution of normalized returns (400 stocks)
- portfolio: superposition of stocks
- risk reduction by diversification (no correlations yet!): returns are more normally distributed, market risk reduced by approx. 50 percent
Correlations

- stocks highly correlated to overall market
- risk reduction by diversification (with correlations): unsystematic risk can be removed, systematic risk (overall market) remains
Market risk versus Credit risk

What’s different for credits?
Zero-coupon bond

\( t = 0 \)

\[
\begin{array}{ccc}
\text{Creditor} & \xrightarrow{\text{Principal}} & \text{Obligor} \\
\end{array}
\]

\( t = T \)

\[
\begin{array}{ccc}
\text{Creditor} & \xleftarrow{\text{Face value}} & \text{Obligor} \\
\end{array}
\]

- **principal**: borrowed amount
- **face value** \( F \): borrowed amount + interest + risk compensation
- **credit contract with simplest cash-flow**
Defaults and Losses

▶ default occurs if the obligor fails to repay

⇒ loss between 0 and face value $F$

▶ possible losses have to be priced into credit contract

▶ correlations are important to evaluate the risk of a credit portfolio
Defaults and Losses

- default occurs if the obligor fails to repay

  ⇒ loss between 0 and face value $F$

- possible losses have to be priced into credit contract

- correlations are important to evaluate the risk of a credit portfolio

- statistical modeling needed

- reduced form models versus structural models
Reduced form models

- macroscopic approach

- different aspects (observables) are modelled independently
  - default events as point process
  - recovery rates modelled independently
  - correlations e.g. as network model
Reduced form models

- macroscopic approach

- different aspects (observables) are modelled independently
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- goal: describe empirical statistical properties and market prices for credit products by callibrating with credit products

- problem: the market may be wrong!
Structural models

- microscopic approach
- dynamical description of risk factors $V_k(t), \ k = 1, \ldots, K$
- default occurs if asset value $V_k(T)$ falls below face value $F_k$
- then the (normalized) loss is $L_k = \frac{F_k - V_k(T)}{F_k}$
Structural models

- microscopic approach
- dynamical description of risk factors $V_k(t), \ k = 1, \ldots, K$
- default occurs if asset value $V_k(T)$ falls below face value $F_k$
- then the (normalized) loss is $L_k = \frac{F_k - V_k(T)}{F_k}$
- e.g. credits with stock portfolio or houses as securities

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Modeling credit risk
A model with jumps and correlations

\[
\frac{dV_k}{V_k} = \mu_k \, dt + \sigma_k \varepsilon_k \sqrt{dt} + dJ_k
\]

Geometric Brownian motion with

- deterministic term \( \mu_k \, dt \)
- diffusion term \( \sigma_k \varepsilon_k \sqrt{dt} \)
- jump term \( dJ_k \), governed by a Poisson process
- \( K \) risk elements \( V_k = V_k(t), \ k = 1, \ldots, K \)
Jump process and return distribution

jumps yield heavy tails in the price and return distributions
Jumps as Poisson process

- we model jumps by Poisson process with intensity $\lambda$
- probability for $n$ jumps between 0 and $t$:
  $$p_n(t) = \frac{(\lambda t)^n}{n!} \exp(-\lambda t)$$
- largest negative jump: -100% of $V(t)$
- we choose shifted log-normal distribution for jump size $\Lambda$
  $$\ln(\Lambda + 1) \sim N(\mu_J + 1, \sigma_J)$$
Correlate $K$ risk elements: one-factor model

- $\varepsilon_k$ is random variable for company $k$
- $\eta$ is common random variable within one branch
- correlated diffusion, uncorrelated jumps:

\[
\frac{dV_k}{V_k} = \mu_k \, dt + \left( \sqrt{1 - c \varepsilon_k} + \sqrt{c \eta} \right) \sigma_k \sqrt{dt} + dJ_k
\]

- add influence of market as a whole

\[
\frac{dV_k}{V_k} = \mu_k \, dt + \left( \sqrt{1 - c \varepsilon_k} + \sqrt{c \eta} \right) \sigma_k \sqrt{dt} + dJ_k + dJ_m
\]
Loss distribution
Individual losses

- normalized loss: $L_k = \frac{F_k - V_k(T)}{F_k}$
- default probability: $P_{D,k} = \int_0^{F_k} p_k(V_k(T)) \, dV_k(T)$
- truncate distribution $p_k(V_k(T)) \rightarrow p_k(L_k)$
Default event

- default indicator

\[ I_k = \begin{cases} 
1, & \text{if } V_k(T) < F_k \quad \text{(default)} \\
0, & \text{if } V_k(T) > F_k \quad \text{(no default)} 
\end{cases} \]

- indicator distribution

\[ \tilde{p}_k(I_k) = (1 - P_{D,k})\delta(I_k) + P_{D,k}\delta(I_k - 1) \]
Portfolio loss distribution

- Portfolio loss: \( L - \frac{1}{K} \sum_{k=1}^{K} L_k I_k \)

- Loss distribution

\[
p(L) = \int_{-\infty}^{+\infty} dI_1 \tilde{p}_1(I_1) \cdots \int_{-\infty}^{+\infty} dI_K \tilde{p}_K(I_K) \int_{0}^{1} dL_1 p_1(L_1) \cdots \int_{0}^{1} dL_K p_K(L_K) \\
\times \delta \left( L - \frac{1}{K} \sum_{k=1}^{K} L_k I_k \right)
\]

- Special case \( K = 1 \) yields: \( p(L) = (1 - P_{D,1}) \delta(L) + P_{D,1} p_1(L) \)
Large portfolios

Real portfolios comprise several hundred or more individual contracts $\rightarrow K$ is large.

Central Limit Theorem: For very large $K$, portfolio loss distribution $p(L)$ must become Gaussian.

Question: how large is “very large”? 
Distribution of credit losses

- **portfolio loss** is arithmetic mean of individual losses
- mean of loss distribution is called **expected loss** (EL)
- standard deviation is called **unexpected loss** (UL)
- kurtosis excess (KE) to measure heavy tails: \( \gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 \)
Simplified model — no jumps, no correlations

- homogenous portfolio
- analytical approximations
- check Monte-Carlo results
Simplified model — no jumps, no correlations

- homogenous portfolio
- analytical approximations
- check Monte-Carlo results
- slow convergence to Gaussian for large portfolio
- \( K = 1000 \) not yet Gaussian CLT–limit
- kurtosis excess of uncorrelated portfolios scales as \( 1/K \)
Numerical simulations
Numerical simulations: influence of jumps, no correlations

- diffusion and jumps compete
- KE has maximum, but scales as $1/K$
Numerical simulations: influence of correlations, no jumps

- correlation coefficient \( c = 0.5 \)
- transition from uncorrelated to fully correlated

\[ c = 0.5 \]
Numerical simulations: influence of correlations, no jumps

- standard deviation decreases
- bad measure for credit risk!
- diversification does not reduce the risk

\[ c = 0.5 \]
Numerical simulations: influence of correlations, no jumps

- correlation coefficient $c = 0.2$
- transition from uncorrelated to fully correlated

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Numerical simulations: jumps and correlations

- correlated jump-diffusion
- one-branch correlations
- $c = 0.5$
- tail behavior stays similar with increasing $K$
Random matrix approach
Quantum Chaos

statistical nuclear physics

universal in a huge variety of systems: nuclei, atoms, molecules, disordered systems, lattice gauge quantum chromodynamics, elasticity, electrodynamics

"second ergodicity": spectral average = ensemble average

→ random matrix theory
Price distribution at maturity

Brownian motion, \( V = (V_1(T), \ldots, V_K(T)) \), price distribution

\[
p^{(mv)}(V, \Sigma) = \frac{1}{\sqrt{2\pi T^K}} \frac{1}{\sqrt{\det \Sigma}} \exp \left( -\frac{1}{2T} (V - \mu T)\Sigma^{-1}(V - \mu T) \right)
\]

covariance matrix \( \Sigma = \sigma W W^\dagger \sigma \) with fixed \( \sigma = \text{diag}(\sigma_1, \ldots, \sigma_K) \)

assume Gaussian distributed correlation matrix \( W W^\dagger \)

with \( W \) rectangular real \( K \times N \), variance \( 1/N \)

\[
p^{(corr)}(W) = \sqrt{\frac{N}{2\pi}}^{KN} \exp \left( -\frac{N}{2} \text{tr} \; W^\dagger W \right)
\]

average correlation is zero

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Average price distribution

\[ \langle p^{(mv)}(\rho) \rangle = \sqrt{\frac{N^K}{2\pi T}} \frac{2^{1-N/2}}{\Gamma(N/2)} \rho^{N+K-1} \sqrt{\frac{N}{T}} \mathcal{K}_{N-K/2} \left( \rho \sqrt{\frac{N}{T}} \right) \]

with hyperradius \( \rho = \sqrt{\sum_{k=1}^{K} \frac{V_{k}^{2}(T)}{\sigma_{k}^{2}}} \)

easily transferred to geometric Brownian motion
Heavy tailed average distribution

\[ K = 50 \text{ and } N = K, 2K, 5K, 30K \]

\( N \) smaller \( \rightarrow \) stronger correlated \( \rightarrow \) heavier tails
Loss distribution — varying correlation strength

integrate out risk elements, semi–analytical result

homogeneous portfolio $K = 10$ and $N = K, 2K, 10K, 30K$

also here: stronger correlated $\rightarrow$ heavier tails
Loss distribution — varying portfolio sizes

homogeneous portfolios $K = 50, 100$, strongly correlated $N = K$

heavy tails robust!
General conclusions

- correlated jumps lead to extremely fat-tailed distribution
- kurtosis excess (KE) scales as $1/K$ for uncorrelated portfolios
- KE does not scale down well for correlated portfolios, even for low correlation coefficients
- correlations of stocks to market movement typically between 0.4 and 0.6
- other scenarios: houses, cars, etc as security for credits
- ensemble average reveals generic features of loss distributions
- lower bound, because average correlation is zero
Conclusions in view of the present credit crisis

- credit contracts with high default probability, e.g. houses as securities
- credit institutes resold the risk of credit portfolios, grouped by credit rating
- lower ratings $\Rightarrow$ higher risk and higher potential return
- problems:
  - rating agencies rated way too high
  - effect of correlations underestimated
  - benefit of diversification overestimated

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both ranked for several months among the top–ten new credit risk papers on www.defaultrisk.com