Random Matrices, Quantum Chaos
and Open Quantum Systems

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Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Qubit = quantum bit; $N = 2$

$$|\psi\rangle = \cos\frac{\psi}{2}|1\rangle + e^{i\phi} \sin\frac{\psi}{2}|0\rangle$$

Bloch sphere of $N = 2$ pure states

Space of pure states for an arbitrary $N$:

a complex projective space $\mathbb{C}P^{N-1}$ of $2N - 2$ real dimensions.
Unitary evolution

Fubini-Study distance in $\mathbb{C}P^{N-1}$

$$D_{FS}(|\psi\rangle, |\varphi\rangle) := \arccos |\langle \psi | \varphi \rangle|$$

Unitary evolution

Let $U = \exp(iHt)$. Then $|\psi'\rangle = U|\psi\rangle$.

Since $|\langle \psi | \varphi \rangle|^2 = |\langle \psi | U^\dagger U | \varphi \rangle|^2$ any unitary evolution is a rotation in $\mathbb{C}P^{N-1}$

hence it is an isometry (with respect to any standard distance!)

Classical limit: what happened for large $N$?

How an isometry may lead to classically chaotic dynamics?

The limits $t \to \infty$ and $N \to \infty$ do not commute.
'Quantum chaology': analogues of classically chaotic systems

Quantum analogues of classically chaotic dynamical systems can be described by random matrices

a). autonomous systems – Hamiltonians:

Gaussian ensembles of random Hermitian matrices, (GOE, GUE, GSE)

b). periodic systems – evolution operators:

Dyson circular ensembles of random unitary matrices, (COE, CUE, CSE)

Universality classes

Depending on the symmetry properties of the system one uses ensembles form orthogonal ($\beta = 1$); unitary ($\beta = 2$) and symplectic ($\beta = 4$) ensembles.

The exponent $\beta$ determines the level repulsion, $P(s) \sim s^\beta$ for $s \to 0$ where $s$ stands for the (normalised) level spacing, $s = \phi_{i+1} - \phi_i$.

see e.g. F. Haake, *Quantum Signatures of Chaos*
Set $\mathcal{M}_N$ of all mixed states of size $N$

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \text{Tr}\rho = 1\}$$

element: $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - Bloch ball with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and convex:

$$\rho = \sum_i a_i |\psi_i\rangle \langle \psi_i|$$

where $a_i \geq 0$ and $\sum_i a_i = 1$.

It has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$.

How the set of all $N = 3$ mixed states looks like?

An 8 dimensional convex set with only 4 dimensional subset of pure (extremal) states, which belong to its 7–dim boundary
Quantum maps

Quantum operation: linear, completely positive trace preserving map

\[ \Phi: \mathcal{M}_2 \rightarrow \mathcal{M}_2 \]

**positivity:** \( \Phi(\rho) \geq 0, \quad \forall \rho \in \mathcal{M}_N \)

**complete positivity:** \( [\Phi \otimes \mathbb{1}_K](\sigma) \geq 0, \quad \forall \sigma \in \mathcal{M}_{KN} \) and \( K = 2, 3, \ldots \)

Enviromental form (open system !)

\[
\rho' = \Phi(\rho) = \text{Tr}_E[U (\rho \otimes \omega_E) U^\dagger].
\]

where \( \omega_E \) is an initial state of the environment while \( UU^\dagger = \mathbb{1} \).

Kraus form

\[
\rho' = \Phi(\rho) = \sum_i A_i \rho A_i^\dagger,
\]

where the Kraus operators satisfy

\[
\sum_i A_i^\dagger A_i = \mathbb{1}, \quad \text{which implies that the trace is preserved.}
\]
Classical probabilistic dynamics & Markov chains

**Stochastic matrices**

**Classical states:** $N$-point probability distribution, $\mathbf{p} = \{p_1, \ldots, p_N\}$, where $p_i \geq 0$ and $\sum_{i=1}^{N} p_i = 1$

**Discrete dynamics:** $p'_i = S_{ij} p_j$, where $S$ is a stochastic matrix of size $N$ and maps the simplex of classical states into itself, $S : \Delta_{N-1} \rightarrow \Delta_{N-1}$.

**Frobenius–Perron theorem**

Let $S$ be a stochastic matrix:

a) $S_{ij} \geq 0$ for $i, j = 1, \ldots, N$,

b) $\sum_{i=1}^{N} S_{ij} = 1$ for all $j = 1, \ldots, N$.

Then

i) the spectrum $\{z_i\}_{i=1}^{N}$ of $S$ belongs to the unit disk,

ii) the leading eigenvalue equals unity, $z_1 = 1$,

iii) the corresponding eigenstate $\mathbf{p}_{\text{inv}}$ is invariant, $S\mathbf{p}_{\text{inv}} = \mathbf{p}_{\text{inv}}$. 
Quantum stochastic maps (trace preserving, CP maps)

Superoperator $\Phi : \mathcal{M}_N \rightarrow \mathcal{M}_N$

A quantum operation can be described by a matrix $\Phi$ of size $N^2$,

$$\rho' = \Phi \rho \quad \text{or} \quad \rho'_{\mu\nu} = \Phi_{m\mu n\nu} \rho_{n\nu}.$$

The superoperator $\Phi$ can be expressed in terms of the Kraus operators $A_i$,

$$\Phi = \sum_i A_i \otimes \bar{A}_i.$$

Dynamical Matrix $D$: Sudarshan et al. (1961)

obtained by reshuffling of a 4–index matrix $\Phi$ is Hermitian,

$$D_{\mu\nu} := \Phi_{m\mu n\nu}, \quad \text{so that} \quad D_\Phi = D_\Phi^\dagger =: \Phi^R.$$

Theorem of Choi (1975). A map $\Phi$ is completely positive (CP) if and only if the dynamical matrix $D$ is positive, $D \geq 0$. 
Spectral properties of a superoperator $\Phi$

**Quantum analogue of the Frobenious-Perron theorem**

Let $\Phi$ represent a stochastic quantum map, i.e.

a') $\Phi^R \geq 0$; (Choi theorem)

b') $\text{Tr}_A \Phi^R = 1 \Leftrightarrow \sum_k \Phi_{kk}^{ij} = \delta_{ij}$. (trace preserving condition)

Then

i') the spectrum $\{z_i\}_{i=1}^{N^2}$ of $\Phi$ belongs to the unit disk,

ii') the leading eigenvalue equals unity, $z_1 = 1$, 

iii') the corresponding eigenstate (with $N^2$ components) forms a matrix $\omega$ of size $N$, which is positive, $\omega \geq 0$, normalized, $\text{Tr} \omega = 1$, and is invariant under the action of the map, $\Phi(\omega) = \omega$.

**Classical case**

In the case of a diagonal dynamical matrix, $D_{ij} = d_i \delta_{ij}$ reshaping its diagonal $\{d_i\}$ of length $N^2$ one obtains a matrix of size $N$, where $S_{ij} = D_{jj}$, of size $N$ which is stochastic and recovers the standard F–P theorem.
Exemplary spectra of (typical) superoperators

Spectra of several **random** superoperators $\Phi$ for a) $N = 2$ and b) $N = 3$ contain:

i) the leading eigenvalue $z_1 = 1$ corresponding to the invariant state $\omega$,

ii) real eigenvalues,

iii) complex eigenvalues inside the disk of radius $r = |z_2| \leq 1$. 
Random (classical) stochastic matrices

Ginibre ensemble of complex matrices

Square matrix of size $N$, all elements of which are independent random complex Gaussian variables.

An algorithm to generate $S$ at random:

1) take a matrix $X$ form the complex Ginibre ensemble
2) define the matrix $S$,

$$S_{ij} := \frac{|X_{ij}|^2}{\sum_{i=1}^{N} |X_{ij}|^2},$$

which is stochastic by construction:

each of its columns forms an independent random vector distributed uniformly in the probability simplex $\Delta_{N-1}$. 
Random quantum states

How to generate a mixed quantum state at random?

1) Fix $M \geq 1$ and take a $N \times M$ random complex Ginibre matrix $X$;
2) Write down the positive matrix $Y := XX^\dagger$,
3) Renormalize it to get a random state $\rho$,

$$\rho := \frac{Y}{\text{Tr} Y}. $$

This matrix is positive, $\rho \geq 0$ and normalised, $\text{Tr} \rho = 1$, so it represents a quantum state!

Special case of $M = N$ (square Ginibre matrices)

Then random states are distributed uniformly with respect to the Hilbert-Schmidt (flat) measure,
e.g. for $M = N = 2$ random mixed states cover uniformly the interior of the Bloch ball.
Random (quantum) stochastic maps

An algorithm to generate \( \Phi \) at random:

1) Fix \( M \geq 1 \) and take a \( N^2 \times M \) random complex Ginibre matrix \( X \);
2) Find the positive matrix \( Y := \text{Tr}_A XX^\dagger \) and its square root \( \sqrt{Y} \);
3) Write the dynamical matrix (Choi matrix)

\[
D = (\mathbb{1}_N \otimes \frac{1}{\sqrt{Y}})XX^\dagger(\mathbb{1}_N \otimes \frac{1}{\sqrt{Y}}) ;
\]

4) Reshuffle the Choi matrix to obtain the superoperator \( \Phi = D^R \) and use

to produce a random map, \( \rho'_{m\mu} = \Phi_{m\mu}^{m\nu} \rho_{n\nu} \).

Map \( \Phi \) obtained in this way is stochastic!
i.e. \( \Phi \) is completely positive and trace preserving
Random stochastic maps II

Probability distribution for random maps

\[ P(D) \propto \det(D^{M-N^2}) \delta(\text{Tr}_A D - 1) , \]

In the special case \( M = N^2 \) the determinant vanishes, so there are other constraints on the distribution of the random Choi matrix \( D \), besides the partial trace relation, \( \text{Tr}_A D = 1 \).

Interaction with \( M \)-dim. enviroment

1’) Chose a random unitary matrix \( U \) according to the Haar measure on \( U(NM) \)

2’) Construct a random map defining

\[ \rho' = \text{Tr}_M[U(\rho \otimes |\nu\rangle\langle\nu|)U^\dagger] , \]

where \( |\nu\rangle \in \mathcal{H}_M \) is an arbitrary (fixed) state of the environment.
Bloch vector representation of any state $\rho$

$$\rho = \sum_{i=0}^{N^2-1} \tau_i \lambda^i$$

where $\lambda^i$ are generators of SU($N$) such that $\text{tr} \left( \lambda^i \lambda^j \right) = \delta^{ij}$ and $\lambda^0 = \mathbb{I}/\sqrt{N}$ and $\tau_i$ are expansion coefficients. Since $\rho = \rho^\dagger$, the generalized Bloch vector $\vec{\tau} = [\tau_0, \ldots, \tau_{N^2-1}]$ is real.

Stochastic map $\Phi$ in the Bloch representation

The action of the map $\Phi$ can be represented as

$$\tau' = \Phi(\tau) = C\tau + \kappa,$$

where $C$ is a real, asymmetric contraction matrix of size $N^2 - 1$ while $\kappa$ is a translation vector. Thus $\Phi = \begin{bmatrix} 1 & 0 \\ \kappa & C \end{bmatrix}$ and the eigenvalues of $C$ are also eigenvalues of $\Phi$. 
Random maps & random matrices

Full rank, symmetric case $M = N^2$

For large $N$ the measure for $C$ can be described by the real Ginibre ensemble of non-hermitian Gaussian matrices.

Spectral density in the unit disk

The spectrum of $\Phi$ consists of:

i) the leading eigenvalue $z_1 = 1$,
ii) the component at the real axis, the distribution of which is asymptotically given by the step function $P(x) = \frac{1}{2} \Theta(x - 1) \Theta(1 - x)$,
iii) complex eigenvalues, which cover the disk of radius $r = |z_2| \leq 1$ uniformly according to the Girko distribution.

Subleading eigenvalue $r = |z_2|$

The radius $r$ is determined by the trace condition: Since the average $\langle \text{Tr} D^2 \rangle = \langle \text{Tr} \Phi \Phi^\dagger \rangle \approx \text{const}$ then $r \sim 1/N$ so the spectrum of the rescaled matrix $\Phi' := N\Phi$ covers the entire unit disk.
Spectral density for random stochastic maps

Numerical results

a) Distribution of complex eigenvalues of $10^4$ rescaled random operators $\Phi' = N\Phi$ already for $N = 10$ can be approximated by the circle law of Girko.

b) Distribution of real eigenvalues $P(x)$ of $\Phi'$ plotted for $N = 2, 3, 7$ and 14 tends to the step function!
Convergence to equilibrium and decoherence rate

**Average trace distance to invariant state** $\omega = \Phi(\omega)$

$L(t) = \langle \text{Tr} | \Phi^t(\rho_0) - \omega \rangle \psi$, where the average is performed over an ensemble of initially pure random states, $\rho_0 = |\psi \rangle \langle \psi |$. Numerical result confirm an exponential convergence, $L(t) \sim \exp(-\alpha t)$.

![Graph](image)

a) Average trace distance of random pure states to the invariant state of $\Phi$ as a function of time for $N = 4(\bullet), 6(\blacksquare), 8(\star)$.

b) mean convergence rate $\langle \alpha \rangle_\Phi$ scales as $\ln N$ with the system size $N$. 
Comparison with a quantum dynamical system

Generalized quantum baker map with measurements

a) Standard quantisation of Balazs and Voros

\[ B = F_N^\dagger \begin{bmatrix} F_{N/2} & 0 \\ 0 & F_{N/2} \end{bmatrix}, \]

where \( F_N \) denotes the Fourier matrix of size \( N \). Then

\[ \rho' = B \rho_i B^\dagger \]

b) \( M \) measurement operators projecting into orthogonal subspaces

\[ \rho_{i+1} = \sum_{i=1}^{M} P_i \rho' P_i \]

Numerical spectra of superoperator of baker map for \( N = 64 \), a) \( M = 2 \), b) \( M = 8 \)  (master thesis of M. Smaczyński, in preparation).
Concluding Remarks

- **Quantum Chaos:**
  a) in case of closed systems one studies unitary evolution operators and characterizes their spectral properties,
  b) for open, interacting systems one analyzes non–unitary time evolution described by quantum stochastic maps.

- We analyzed spectral properties of quantum stochastic maps and formulated a quantum analogue of the Frobenius-Perron theorem.

- A natural flat measure in the space of quantum operations (stochastic maps) is defined and an algorithm to produce them at random is given.

- For large $N$ random quantum operations can be described by random matrices of the real Ginibre ensemble.

- Sequential action of a fixed random map brings all pure states to the invariant state exponentially fast. The convergence rate scales logarithmically with the system size $N$. 