Entanglement and random quantum states

Marko Žnidarič

Department of Physics
Faculty of Mathematics and Physics
University of Ljubljana, Slovenia

Maribor, July 2008
Outline

1. Quantum entanglement
2. Entanglement of random pure states
3. Generating random pure states
4. Practicality of entanglement detection
5. Role of generic initial states
Quantum information

Quantum feats:

- **Quantum secure communication**  
  (no entanglement required, just no cloning)

- **Teleportation**  
  (entanglement needed, e.g., EPR state)

- **Quantum computation**  
  (sufficient entanglement necessary (but not sufficient), else efficient classical simulation possible)
Hilbert space

\[ \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \]

Usually we talk about qubits as basic units:

- system with two levels \( |0\rangle \) and \( |1\rangle \); 2 dimensional Hilbert space:
  - spin \( \frac{1}{2} \) particle (electron): two orthogonal states are spin up and spin down
  - photon polarization: two linear (circular) polarizations
  - two energy states of an ion

- Whole system of \( n \) qubits: Hilbert space is \( \mathcal{H} = \mathcal{H}_i \otimes^n \), 
  \( \dim(\mathcal{H}) = 2^n \) (exponential in \( n \))

- Elements from Hilbert space in computational basis
  \( |01 \ldots 1\rangle = |0\rangle \otimes |1\rangle \otimes \ldots \otimes |1\rangle \).
Definition of a separable state:

- **Definition of a separable state:**

**Pure states**

$$|\psi\rangle = |\psi^A\rangle \otimes |\psi^B\rangle$$

**Mixed states (density matrices)**

$$\rho = \sum_i p_i |\psi_i^A\rangle \langle \psi_i^A| \otimes |\psi_i^B\rangle \langle \psi_i^B|$$

$p_i > 0$ and $\sum_i p_i = 1$ ($|\psi_i^{A,B}\rangle$ need not be orthogonal)
Entangled states

A state is **entangled** if it is not separable.

Basis states $|0\rangle$ and $|1\rangle$ (aka. quantum bits - **qubits**).

- Pure entangled state of two qubits:

  $$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

  $$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Bell or EPR states.
Random quantum states - motivation

Analogy:
(classical) random numbers \iff (quantum) random states

Why study?
- They are generic (typical state).
- Complex quantum system - random state during evolution (quantum chaos).
- Shared entangled state is a useful resource!
  (state with a large Schmidt rank, e.g., random, maximally entangled...)
Random quantum states (def.)

Random pure states - definition

Several possibilities to define random $|\psi\rangle = \sum_i c_i |i\rangle$:

- $c_i$ are random Gaussian complex numbers
- $|\psi\rangle$ is eigenvector of a random Hermitian matrix
- $|\psi\rangle$ is a column of a random unitary matrix
  - unique unitarily invariant Haar measure

Questions

1. What are their entanglement properties?
2. How to generate them?
Entanglement of pure states

Schmidt decomposition:

\[ |\psi\rangle = \sum_{i=0}^{N_A-1} \sqrt{\lambda_i} |w_i^A\rangle \otimes |w_i^B\rangle. \]

- \( |w_i^A\rangle \) and \( |w_i^B\rangle \) are orthonormal
- \( \lambda_i \) are eigenvalues of the reduced \( \rho_A = \text{tr}_B |\psi\rangle\langle\psi| \)

- For mixed states it is hard to quantify entanglement
- For pure states easy: all \( \lambda_i \) completely characterize it
  - if all equal, \( \lambda_i = \frac{1}{N_A} \), “the most” entangled state; in 2 \( \times \) 2 this is for instance EPR state

Can we calculate \( \lambda_i \) for random pure states?
To calculate average $\langle \lambda_i \rangle$ (average over random states) in the limit $N_A \to \infty$ use Marčenko-Pastur for the density of eigenvalues (Žnidarič, JPA 40 F105 ’07)

- $w = 1/2^{2r} = N_A/N_B$ (bipartition to $n/2 - r$ and $n/2 + r$ spins)
- $w \ll 1 \implies \rho_A \approx \frac{1}{N_A} \mathbb{1}$
- Deviations from $\lambda_i = 1/N_A$ are $\sim \frac{2}{N_A} \sqrt{w}$, i.e., exponentially small in the number of “particles” in $\mathcal{H}_B$. 

![Graph showing eigenvalues for random states](image)
How to generate random states?

- In principle we need $2N - 1$ parameters for random $|\psi\rangle$ (too many). They are generic, but are they physical?
- We want a method that is polynomial in $n = \log(N)$

**Example**

- start with a non-random $|\psi\rangle$, e.g., $|00 \ldots 0\rangle$
- at each step apply a random 2-qubit gate to a random pair of qubits

How many steps do we need?
**Number of steps**

Number of steps until all eigenvalues $\approx 1/N_A$, purity

$$l = \text{tr}_A \rho_A^2 \approx 1/N_A$$

($|\psi\rangle$ is as entangled as a typical random state)

**Single step analysis**

- expand $\rho = |\psi\rangle\langle\psi|$ over Pauli basis,

$$\rho(c_i) = \sum_i c_i \sigma^{i_1} \otimes \sigma^{i_2} \otimes \cdots \otimes \sigma^{i_n}$$

- $\sigma^{i_j} \in \{1, \sigma^x, \sigma^y, \sigma^z\}$, matrix basis for $U(2)$.

- after one step you get $\rho'(c'_i) = U \rho(c_i) U^\dagger$

- to calculate purity we need $c_i^2$

- it turns out that $(c'_i)^2$ depend **linearly** on $(c_i)^2$ (no $c_i c_j$ terms)!

- **Markov chain**, $(c')^2 = M \cdot c^2$ (Oliveira, Dahlstein, Plenio, PRL 98, 130502 (07))
Markov chain

Markov chain only if two-qubit gate preserves Pauli matrices \((W\sigma^\alpha W^\dagger = \sigma^\beta)\)
- dimension of \(M\) is \(4^n\)
- What is the gap \(\Delta\)? \(\longrightarrow\) number of needed steps
- Is the chain rapidly mixing, \(i.e., \Delta \sim 1/\text{poly}(n)\)?

- Analytical estimate for \(W = \text{CNOT}\) and random \(i - j\) coupling: \(\Delta > \frac{4}{9n(n-1)}\) \((\text{Oliveira et al. (07)})\)
- Numerics gives \((\text{Žnidarič, PRA 76, 012318 (07)})\) \(\Delta \asymp 1.6/n\).
Analytical solution

Space of n “qudits”, e.g., each site 4 states (Pauli matrices).

\[ M = \frac{1}{n} \sum_{i}^{n} T_{i,i+1} \otimes 1 \]

T transition matrix for two “qudits” \((4^2 \times 4^2)\) and \(U(4)\) gate,

\[ T = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & \frac{1}{15} & \cdots & \frac{1}{15} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{1}{15} & \cdots & \frac{1}{15}
\end{pmatrix}. \]

Calculate the gap \(\Delta!\)
Analytical solution (cont.)

Markov chain on $4^n$ equivalent to spin chain on $2^n$

$U(4)$ and nearest neighbor coupling – $XY$ model:

$$h_{XY} = \frac{1 + \gamma}{2} \sigma_i^x \sigma_j^x + \frac{1 - \gamma}{2} \sigma_i^y \sigma_j^y + h(\frac{1}{2} \sigma_i^z + \frac{1}{2} \sigma_j^z).$$

$U(4)$ and all-all coupling – Lipkin-Meshkov-Glick:

$$h S_z + J_x S_x^2 + J_y S_y^2$$

CNOT and $XY$ gates – $XYZ$ model

Analytical gap $\Delta \sim \frac{1}{n}$
Entanglement and classicality

Question

1. Why is there no observable entanglement in macro-world?

Classical irreversibility:

- practical issues of reversibility: almost impossible to reverse
- role of initial conditions: for most entropy increases

picture from R. Penrose
Practicality

Random states are quantum

- almost maximally entangled, von Neumann entropy $S \approx \frac{n}{2}$
  
  random states are very entangled - very quantum

...are classical

- in classical limit ($N \to \infty$) random states mimic microcanonical density
- quantum expectation value in a random state is close to the classical average
Paradox

How come?

Resolution:

- von Neumann entropy does not tell everything!
- Entanglement hidden in many degrees of freedom, e.g., Schmidt coefficients are $\sim 1/\sqrt{N_A}$ - exponentially small.
- Difficult to detect!

For all practical purposes classical.
Entanglement Witness

Definition

- If \( \text{tr}(\rho_{\text{sep}} W) > 0 \) for all separable \( \rho_{\text{sep}} \) and \( \text{tr}(\rho_{\text{ent}} W) < 0 \) for at least one entangled \( \rho_{\text{ent}} \), \( W \) is an entanglement witness. It detects entanglement of \( \rho_{\text{ent}} \).
- In general different \( W \) for different \( \rho_{\text{ent}} \).

Decomposable EW

Especially simple are decomposable EW:

\[
W = P + Q^{T_B}, \quad P, Q \geq 0
\]

- \( Q^{T_B} \) is partial transposition with respect to subspace B
- D-EW are equivalent to PPT criterion
Example of $W$

Example

- Take for $Q$ a projector, $Q = |GHZ\rangle\langle GHZ|$ with $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and $P = 0$.

- Subsystem B is last qubit, $W = Q^{TB}$,
  $W = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111| + |001\rangle\langle 110| + |110\rangle\langle 001|)$.

- $W$ has one negative eigenvalue with the eigenvector $|\psi\rangle = \frac{1}{2}(|001\rangle - |110\rangle)$.

- $\langle \psi | W | \psi \rangle = -\frac{1}{2}$. Detects entanglement of $|\psi\rangle$.

- $\langle GHZ | W | GHZ \rangle = \frac{1}{2}$. Does not detect entanglement of $|GHZ\rangle$. 
**Results** (M.Ž., T.Prosen, G.Benenti and G.Casati, JPA 40, 13787 (2007))

- **Large random states almost classical.**
- **Random** $W$ (unknown $\rho$): Gaussian $p(w)$,
  \[ \text{tr}(W\rho) \sim -1/N_A^2 \]
  - \[ P(w < 0) = (1 - \text{erf}(1/\sqrt{2}))/2 \approx 0.16 \]
  - Mixing $k$ states, $\rho \sim \sum_{i=1}^{k} |\psi_i\rangle\langle\psi_i|$, 
    \[ P(w < 0) = (1 - \text{erf}(\sqrt{k}/2))/2 \approx \frac{1}{\sqrt{k}}e^{-k/2} \]
- **Optimal** $W$ (known $\rho$): $\text{tr}(W\rho) = -|\lambda_{\text{min}}(\rho^{TB})|$ 
  - Pure state ($k = 1$): $\lambda_{\text{min}} = -4/N_A$ 
  - Large $k \gg 1$:
    - $\lambda_{\text{min}} \sim -1/N_A^2$ 
    - $k > k^* \approx 4N_A^2$: $\lambda_{\text{min}} > 0$
Initial conditions

Setting

- Large $n$ qubit quantum system
- Start in generic separable state (no entanglement)
- Evolve with some Hamiltonian
- What is entanglement of smaller subsystem (two qubits)

How much entanglement, for how long...?

We would “like” to see: For generic i.c. low entanglement only for short times and regardless of $H$!
Arbitrary $H$ with two-particle coupling $h$. Initial time scale dictated by

$$\lambda_{\text{min}}^{TA} = -|\delta| t + \mathcal{O}(t^2), \quad \delta = \langle \chi_A^\perp \chi_B^\perp | h^{(2)} | \chi_A \chi_B \rangle.$$ 

- n.n. two-body RMT model
- distance between qubits $r$
- universality: almost the same dependence for any $H$
Initial state randomness as a universal source of decoherence

- randomness in initial state
- leads to universal behavior of entanglement between two qubits regardless of the coupling
- entanglement present only for short time and directly coupled qubits
Summary

- Giving Schmidt coefficients completely determines entanglement of pure states – analytical expression
- Generating random bipartite entanglement in $\tau \sim n \ln \frac{1}{\epsilon}$, gap $\Delta \sim 1/n$

No entanglement in systems with many degrees of freedom:
- Practicality: hard to detect because many small Schmidt coefficients
- Generic initial states: entanglement only for short times and directly coupled qubits. Independent of $H$!