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New trends in quantum chaos of generic systems

by

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Let's Face Chaos through Nonlinear Dynamics,

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CONTENTS

(1)

Hamiltonian systems

$$H = H(\vec{q}, \vec{p})$$

$\vec{q} = (q_1, q_2, \cdots, q_N)$
$\vec{p} = (p_1, p_2, \cdots, p_N)$

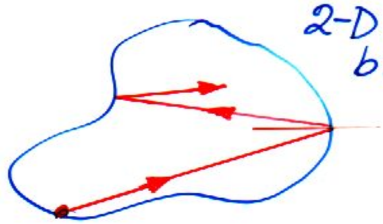
$$\begin{cases} \dot{\vec{q}} = \dfrac{\partial H}{\partial \vec{p}} \\ \dot{\vec{p}} = -\dfrac{\partial H}{\partial \vec{q}} \end{cases}$$ Hamilton equations

autonomous systems: $E = H(\vec{q}, \vec{p}) = \text{const.}$

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{q})$$

$$m\ddot{\vec{q}} = \dot{\vec{p}} = -\frac{\partial V}{\partial \vec{q}}$$

Newton eqs.



2-D billiards

$$\hat{H} = H(\hat{\vec{q}}, \hat{\vec{p}}), \quad \hat{\vec{q}} = \vec{q}, \quad \hat{\vec{p}} = \frac{\hbar}{i}\frac{\partial}{\partial \vec{q}}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(\vec{q}), \quad \hat{H}\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\Delta\psi + (V(\vec{q}) - E)\psi = 0$$

Schrödinger equation plus boundary conditions

billiards: $\Delta\psi + \dfrac{2m}{\hbar^2}E\psi = 0$

$$\psi|_{\partial B} = 0$$

integrable Hamiltonian systems:

N integrals (constants) of motion exist
N = number of degrees of freedom

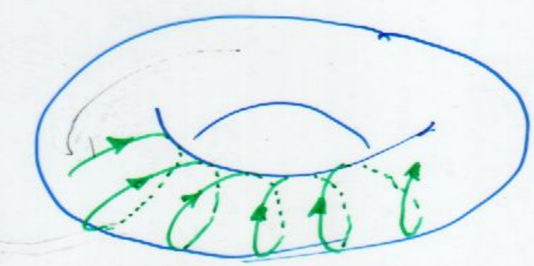
$$A_i = A_i(\vec{q},\vec{p}) = A_i(\vec{q}(t),\vec{p}(t)) = \text{const.}$$

$$i = 1,2,\ldots,N \qquad (A_1 = E = H(\vec{q},\vec{p}))$$

$$\{A_i, A_j\} = \text{Poisson bracket} = 0, \ \forall i,j$$

$$= \frac{\partial A_i}{\partial \vec{q}}\cdot\frac{\partial A_j}{\partial \vec{p}} - \frac{\partial A_i}{\partial \vec{p}}\cdot\frac{\partial A_j}{\partial \vec{q}} \equiv 0$$

Liouville-Arnold theorem:



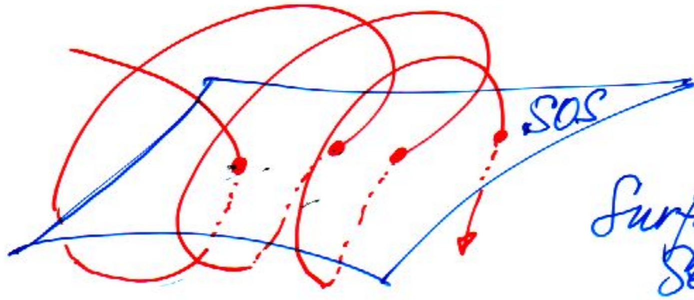
example ($N=2$)

N-dim invariant tori (for all initial conditions)

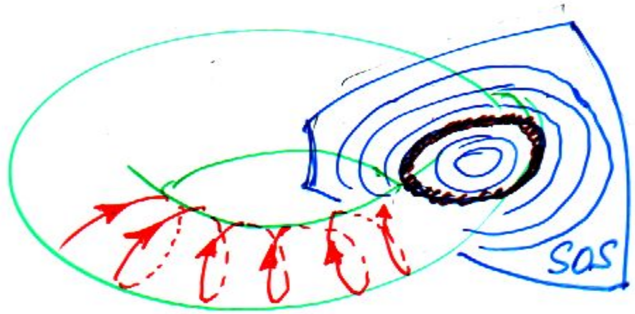
The ergodic systems (fully chaotic):

no integrals of motion except the total energy $E = H(\vec{q},\vec{p})$ = const.

(3)



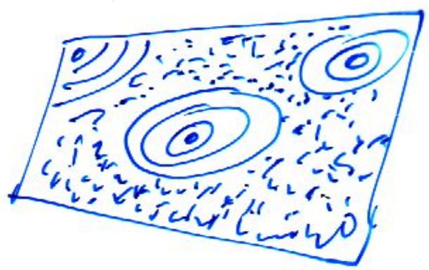
Surface of Section



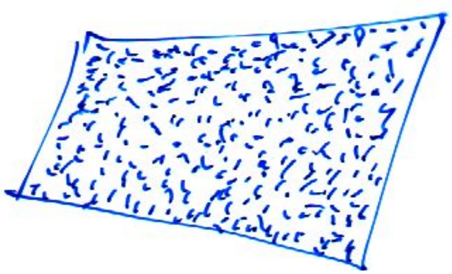
integrable:
N-tori
everywhere

perturb ↓

Kolmogorov
Arnold
Moser



KAM
picture



ergodic
and
chaotic

The Main Assertion of Stationary Quantum Chaos
(Casati, Valz-Gries, Guarneri 1980; Bohigas, Giannoni, Schmit 1984; Percival 1973)

(A1) If the system is classically integrable: **Poissonian spectral statistics**

(A2) If classically fully chaotic (ergodic): **Random Matrix Theory (RMT)** applies

- If there is an antiunitary symmetry, we have GOE statistics
- If there is no antiunitary symmetry, we have GUE statistics

(A3) If of the mixed type, in the deep semiclassical limit: we have no spectral correlations: the spectrum is a **statistically independent superposition of regular and chaotic level sequences**:

$$E(k,L) = \sum_{k_1+k_2+\ldots+k_m=k} \prod_{j=1}^{j=m} E_j(k_j, \mu_j L) \tag{1}$$

$\mu_j =$ relative fraction of phase space volume = relative density of corresponding quantum levels.

$j = 1$ is the Poissonian sequence, $j = 2$ the largest chaotic, $j = 3$ the next largest chaotic etc. Of course: $\mu_1 + \mu_2 + ... + \mu_m = 1$

Special case: The gap probability:

$$E(0, L) = \prod_{j=1}^{j=m} E_j(0, \mu_j L) \tag{2}$$

and remember: $P(S) =$ level spacing distribution $= \frac{d^2 E(0,S)}{dS^2}$

Typically we have just one regular $j = 1$ and one chaotic $j = 2$ sequence:

$$E(k, L) = \sum_{k_1 + k_2 = k} E_{Poisson}(k_1, \mu_1 L) E_{RMT}(k_2, \mu_2 L) \tag{3}$$

(A4) If we are not sufficiently deep in the semiclassical limit (the effective Planck constant is not sufficiently small) we see deviations from PUSC, namely localization and tunneling phenomena, and therefore deviations from **(A3)**

Example of mixed type system: Hydrogen atom in strong magnetic field

Example of mixed type system: Hydrogen atom in strong magnetic field

(Diamagnetic Kepler Problem)

$$H = \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} + \frac{eL_z}{2m_e c}|\mathbf{B}| + \frac{e^2\mathbf{B}^2}{8m_e c^2}\rho^2$$

$\mathbf{B}$ = magnetic field strength vector pointing in z-direction

$r = \sqrt{x^2+y^2+z^2}$ = spherical radius, $\rho = \sqrt{x^2+y^2}$ = axial radius

L_z = z-component of angular momentum = conserved quantity

Characteristic field strength: $B_0 = \frac{m_e^2 e^3 c}{\hbar^2} = 2.35 \times 10^9$ Gauss $= 2.35 \times 10^5$ Tesla

Rough qualitative criterion for global chaos: magnetic force $\approx$ Coulomb force

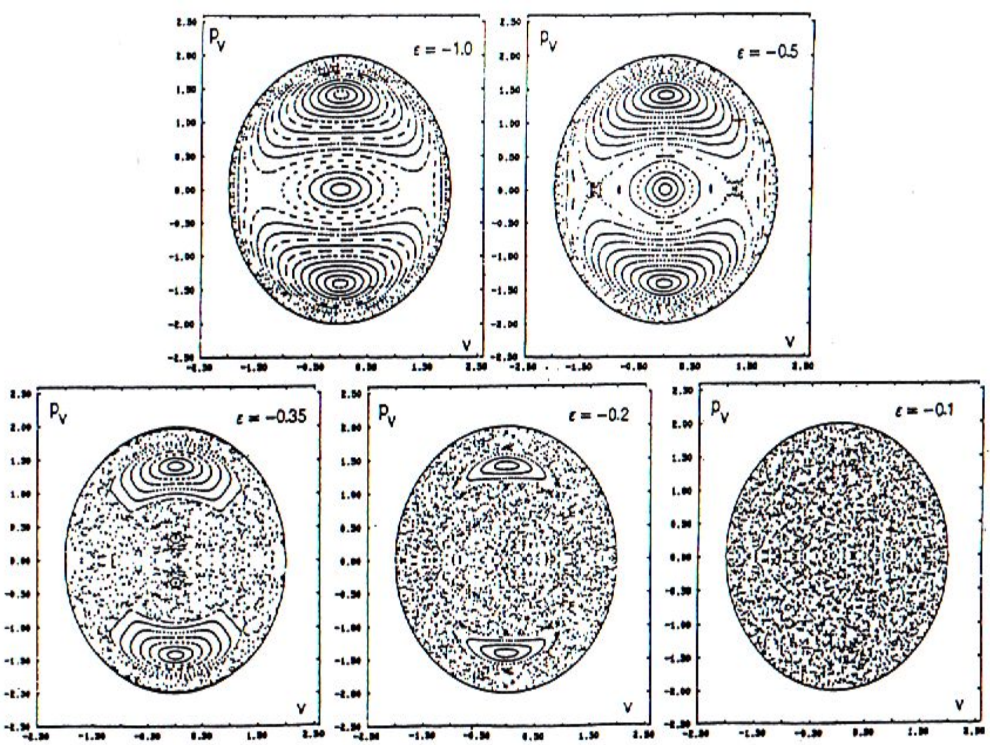


Fig. III-9. Poincaré surfaces of section $\Sigma(v, p_v; u=0)$ at different scaled energies (corresponding to increasing diamagnetic strength). The elliptic fixed point at the origin corresponds to the straight-line orbit I_∞, the other two fixed points to the straight-line orbit I_1.

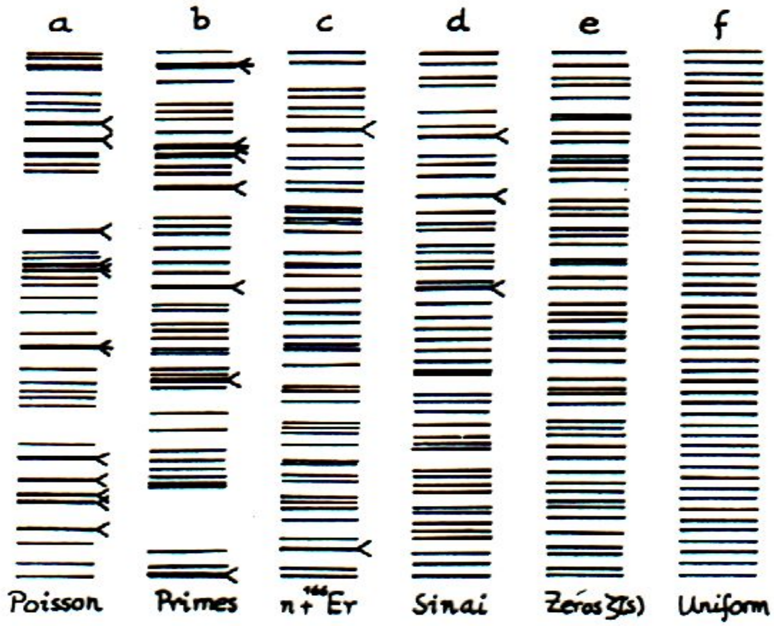
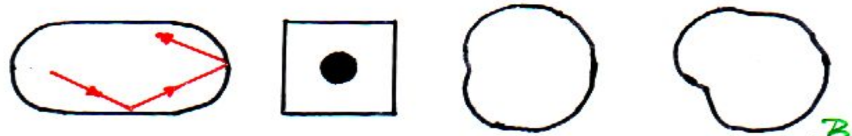


Fig.I.8 - Segments of "spectra", each containing 50 levels. The "arrowheads" mark the occurrence of pairs of levels with spacings smaller than 1/4. See text for further explanation.

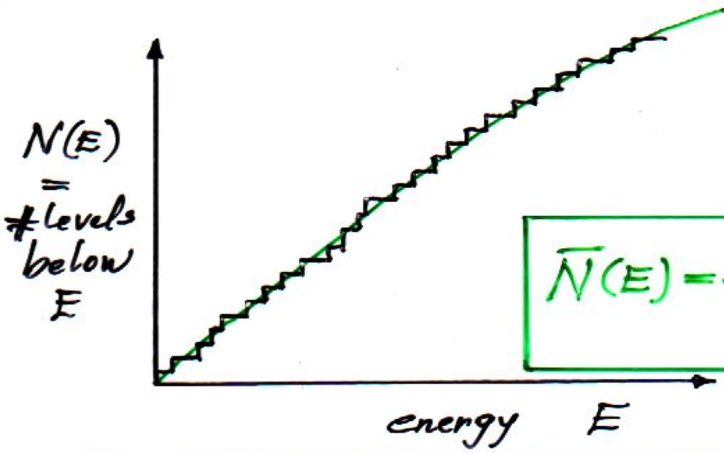
Bohigas and Giannoni 1984

Example: 2-dim billiard systems



Bunimovich (stadium) | Sinai | $w = z + \lambda z^2$ (Robnik 1983) | Africa (Berry & Robnik 1986)

$w = z + Bz^2 + Ce^{i\phi}z^3$, $B = 0.2$, $C = 0.2$, $\phi = \pi/3$



$$\Delta\psi + E\psi = 0$$
$$\psi|_{boundary} = 0$$

$$\bar{N}(E) = \frac{\mathcal{A}E}{4\pi} - \frac{\mathcal{L}\sqrt{E}}{4\pi} + \frac{1}{6}$$
Weyl formula

$$N(E) = \bar{N}(E) + \tilde{N}(E)$$

Unfolding procedure: $x \equiv \bar{N}(E) = x(E)$

$$\mathcal{N}(x) = x + \tilde{\mathcal{N}}(x)$$

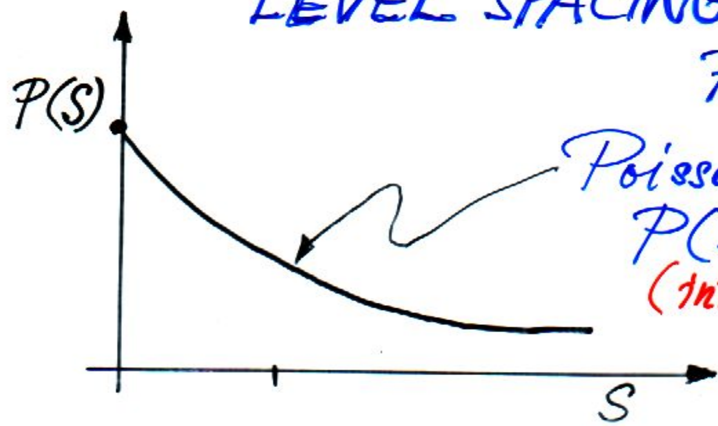
- level spacings distribution: $P(S)$

$P(S)dS =$ Prob. level spacing $S \in [S, S+dS]$

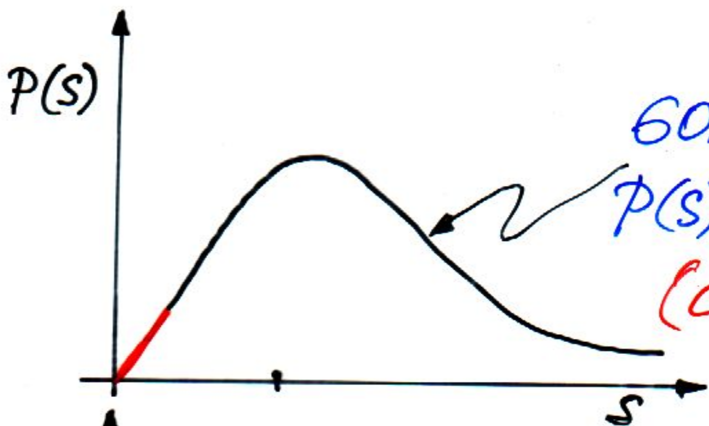
normalized: $\int_0^\infty P(x)dx = 1$, $\int_0^\infty x\,P(x)dx = 1$

cumulative: $W(S) = \int_0^S P(x)\,dx$

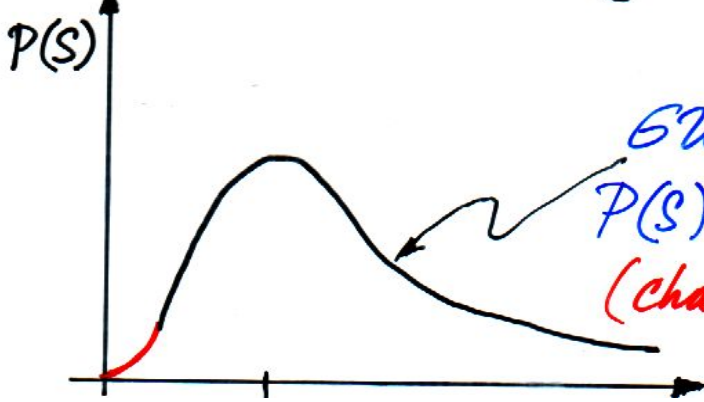
ENERGY (NEAREST NEIGHBOUR) LEVEL SPACING DISTRIBUTION $P(S)$



Poisson
$$P(S) = e^{-S}$$
(integrability)



GOE ≈ Wigner
$$P(S) = \frac{\pi S}{2} e^{-\frac{\pi S^2}{4}}$$
(chaos & A.U.S.)



GUE ≈ Wigner
$$P(S) = \frac{32 S^2}{\pi^2} e^{-\frac{4S^2}{\pi}}$$
(chaos, <u>no</u> A.U.S.)

2D GOE and GUE of random matrices:

Quite generally, for a Hermitian matrix $\begin{pmatrix} x & y+iz \\ y-iz & -x \end{pmatrix}$ with x, y, z real

the eigenvalue $\lambda = \pm\sqrt{x^2+y^2+z^2}$ and level spacing
$S = \lambda_1 - \lambda_2 = 2\sqrt{x^2+y^2+z^2}$

The level spacing distribution is

$$P(S) = \int_{R^3} dx\; dy\; dz\; g_x(x) g_y(y) g_z(z) \delta(S - 2\sqrt{x^2+y^2+z^2}) \qquad (4)$$

which is equivalent to 2D GOE/GUE when $g_x(u) = g_y(u) = g_z(u) = \frac{1}{\sigma\sqrt{\pi}} \exp(-\frac{u^2}{\sigma^2})$
and after normalization to $< S >= 1$

- **2D GUE** $P(S) = \frac{32S^2}{\pi^2} \exp(-\frac{4S^2}{\pi})$ Quadratic level repulsion

- **2D GOE** $g_z(u) = \delta(u)$ and $P(S) = \frac{\pi S}{2} \exp(-\frac{\pi S^2}{4})$ Linear level repulsion

There is no free parameter: Universality

2. Principle of Uniform Semiclassical Condensation (PUSC) of Wigner functions of eigenstates (Percival 1973, Berry 1977, Shnirelman 1979, Voros 1979, Robnik 1987-1998)

We study the structure of eigenstates in "quantum phase space": **The Wigner functions of eigenstates** (they are real valued but **not positive definite**):

Definition: $W_n(\mathbf{q},\mathbf{p}) = \frac{1}{(2\pi\hbar)^N}\int d^N\mathbf{X}\exp\left(-\frac{i}{\hbar}\mathbf{p}.\mathbf{X}\right)\psi_n(\mathbf{q}-\frac{\mathbf{X}}{2})\psi_n^*(\mathbf{q}+\frac{\mathbf{X}}{2})$

$(P1) \quad \int W_n(\mathbf{q},\mathbf{p})d^N\mathbf{p} = |\psi_n(\mathbf{q})|^2$

$(P2) \quad \int W_n(\mathbf{q},\mathbf{p})d^N\mathbf{q} = |\phi_n(\mathbf{p})|^2$

$(P3) \quad \int W_n(\mathbf{q},\mathbf{p})d^N\mathbf{q}\, d^N\mathbf{p} = 1$

$(P4) \quad (2\pi\hbar)^N\int d^N\mathbf{q}\, d^N\mathbf{p} W_n(\mathbf{q},\mathbf{p})W_m(\mathbf{q},\mathbf{p}) = \delta_{nm}$

$(P5) \quad |W_n(\mathbf{q},\mathbf{p})| \le \frac{1}{(\pi\hbar)^N}$ (Baker 1958)

$(P6 = P4) \quad \int W_n^2(\mathbf{q},\mathbf{p})d^N\mathbf{q}\, d^N\mathbf{p} = \frac{1}{(2\pi\hbar)^N}$

$(P7) \quad \hbar \to 0: \quad W_n(\mathbf{q},\mathbf{p}) \to (2\pi\hbar)^N W_n^2(\mathbf{q},\mathbf{p}) > 0$

In the semiclassical limit the Wigner functions condense on an element of phase space of volume size $(2\pi\hbar)^N$ (elementary quantum Planck cell) and become positive definite there.

Principle of Uniform Semiclassical Condensation (PUSC)

Wigner fun. $W_n(\mathbf{q},\mathbf{p})$ condenses uniformly on a classically invariant component:

(C1) invariant N-torus (integrable or KAM): $W_n(\mathbf{q},\mathbf{p}) = \frac{1}{(2\pi)^N}\delta\left(\mathbf{I}(\mathbf{q},\mathbf{p}) - \mathbf{I_n}\right)$

(C2) uniform on topologically transitive chaotic region:

$$W_n(\mathbf{q},\mathbf{p}) = \frac{\delta(E_n - H(\mathbf{q},\mathbf{p}))\ \chi_\omega(\mathbf{q},\mathbf{p})}{\int d^N\mathbf{q}\ d^N\mathbf{p}\ \delta(E_n - H(\mathbf{q},\mathbf{p}))\ \chi_\omega(\mathbf{q},\mathbf{p})}$$

where $\chi_\omega(\mathbf{q},\mathbf{p})$ is the characteristic function on the chaotic component indexed by ω

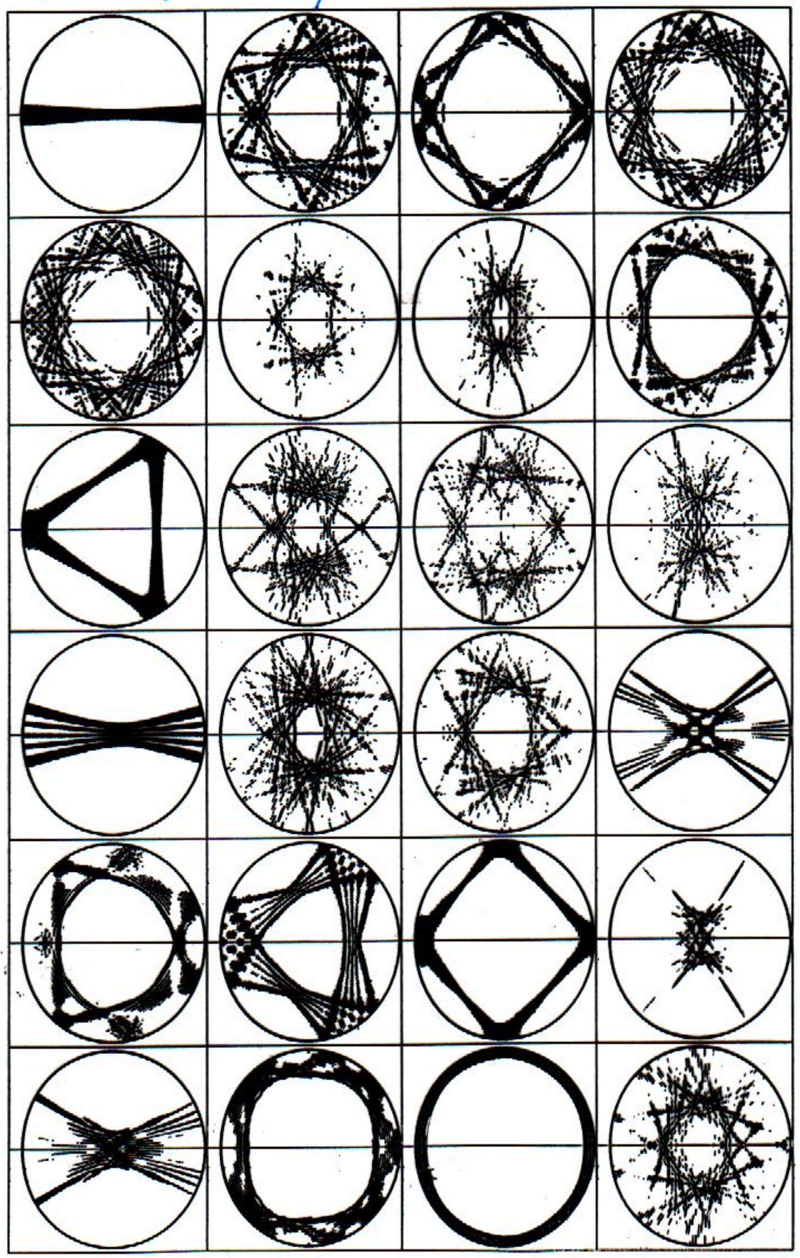
(C3) ergodicity: microcanonical: $W_n(\mathbf{q},\mathbf{p}) = \frac{\delta(E_n - H(\mathbf{q},\mathbf{p}))}{\int d^N\mathbf{q}\ d^N\mathbf{p}\ \delta(E_n - H(\mathbf{q},\mathbf{p})}$

Important: Relative Liouville measure of the classical invariant component:

$$\mu(\omega) = \frac{\int d^N\mathbf{q}\ d^N\mathbf{p}\ \delta(E_n - H(\mathbf{q},\mathbf{p}))\ \chi_\omega(\mathbf{q},\mathbf{p})}{\int d^N\mathbf{q}\ d^N\mathbf{p}\ \delta(E_n - H(\mathbf{q},\mathbf{p}))}$$

$W = z + \lambda z^2$ (Robnik 1983) $\lambda = 0.15$

Figure 1



3. Mixed type systems in the semiclassical limit

3.1 Statistical independence of regular and chaotic level sequences

E(k,L) probabilities (after unfolding!)

- Definition

$E(k,L) =$ probability of having precisely k levels on an interval of length L.

- $< k >= L$

- $E(k=0,L) =$ gap probability (no level in L)

- connection to $P(S)$, $\Sigma(L)$ and $\Delta(L)$:

$P(S) = \frac{d^2E(0,S)}{dS^2}$, $\Sigma(L) = \sum_{k=0}^{\infty}(k-L)^2E(k,L)$ and

$\Delta(L) = \frac{2}{L^4}\int_0^L(L^3 - 2L^2r + r^3)\Sigma(r)dr$

- **Poisson:** $E_{Poisson}(k,L) = \frac{L^k}{k!}e^{-L}$, $P(S) = e^{-S}$, $\Sigma(L) = L$, $\Delta(L) = \frac{L}{15}$.

- **RMT: GOE and GUE** for $k \leq 7$ tables in book of Mehta (1991)

- **RMT: GOE and GUE** for $k \geq 8$ Gaussian approximations:

$E(k,L) \approx \frac{1}{\sqrt{2\pi\alpha(L)}} \exp\left(-\frac{(L-k)^2}{2\alpha(L)}\right)$ where $\alpha(L) = \Sigma(L)$.

The general case of mixed type in the strict semiclassical (”deep”) limit of sufficiently small effective $\hbar$ under the statistical independence assumption:

$$E(k,L) = \sum_{k_1+k_2+...+k_m=k} \prod_{j=1}^{j=m} E_j(k_j, \mu_j L) \quad (5)$$

μ_j= relative fraction of phase space volume = relative density of corresponding quantum levels.

$j = 1$ is the Poissonian sequence, $j = 2$ the largest chaotic, $j = 3$ the next largest chaotic etc. Of course: $\mu_1 + \mu_2 + ... + \mu_m = 1$

Special case: The gap probability: $E(0,L) = \prod_{j=1}^{j=m} E_j(0, \mu_j L)$

and remember: $P(S) =$ level spacing distribution $= \frac{d^2E(0,S)}{dS^2}$

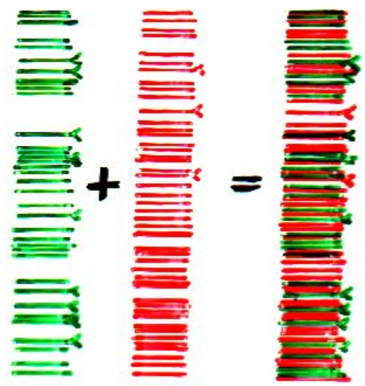
Typically we have just one regular $j=1$ and one chaotic $j=2$ sequence:

$E(k,L) = \sum_{k_1+k_2=k} E_{Poisson}(k_1, \mu_1 L) E_{RMT}(k_2, \mu_2 L)$

How good is this theory at sufficiently small effective $\hbar$?

2. Recent results on the energy level statistics in the transition region between integrability and chaos

fractional volume:
regular regions ϱ_1
chaotic region ϱ_2



+ =

Berry & Robnik 1984 (statistical independence)

Poisson $\langle \Delta E \rangle = 1/\varrho_1$ GOE $\langle \Delta E \rangle = 1/\varrho_2$ $\langle \Delta E \rangle = 1$ $(\varrho_1 + \varrho_2 = 1)$

$$P_1(S) = \varrho_1 e^{-\varrho_1 S}, \quad P_j(S) \cong \frac{\pi \varrho_j^2 S}{2} \exp\left(-\frac{\pi}{4}\varrho_j^2 S^2\right)$$

The answer:

$$P(S) = \frac{d^2 Z}{dS^2}, \quad Z(S) = \prod_{j=1}^{m} \varrho_j Z_j(S)$$

where $Z_j(S) = \int_S^\infty dt\, P_j(t)(t-S)$

$$P_m(S) = \frac{d^2}{dS^2}\left[e^{-\varrho_1 S} \prod_{j=2}^{m} \mathrm{erfc}\left(\frac{\sqrt{\pi}}{2}\varrho_j S\right)\right]$$

$$\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, e^{-t^2}$$

and (as a consequence of the statistical independence)

$$P_m(S=0) = 1 - \sum_{j=2}^{m} \rho_j^2$$

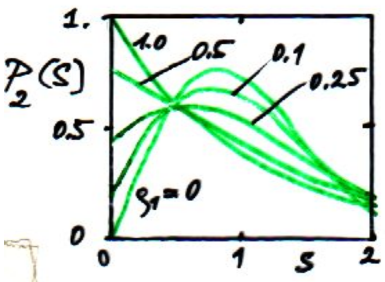
Special case $m=2$:

$$P_2(S,\rho_1) = \rho_1^2 e^{-\rho_1 S} \operatorname{erfc}\left(\frac{\sqrt{\pi}}{2}\rho_2 S\right) + \\ + \left(2\rho_1\rho_2 + \frac{1}{2}\pi\rho_2^3 S\right)\exp\left(-\rho_1 S - \frac{1}{4}\pi\rho_2^2 S^2\right)$$

and

$$P_2(S=0,\rho_1) = 1-\rho_2^2 = \rho_1(2-\rho_1)$$

vanishes only if $\rho_1=0$, $\rho_2=1$



Berry & Robnik 1984

Similarly, upon the assumption of statistical independence:

$$\Delta(L) = \sum_{j=1}^{m}{}' \Delta_j(\rho_j L)$$

(Seligman and Verbaarschot 1985)

Prosen and Robnik 1999

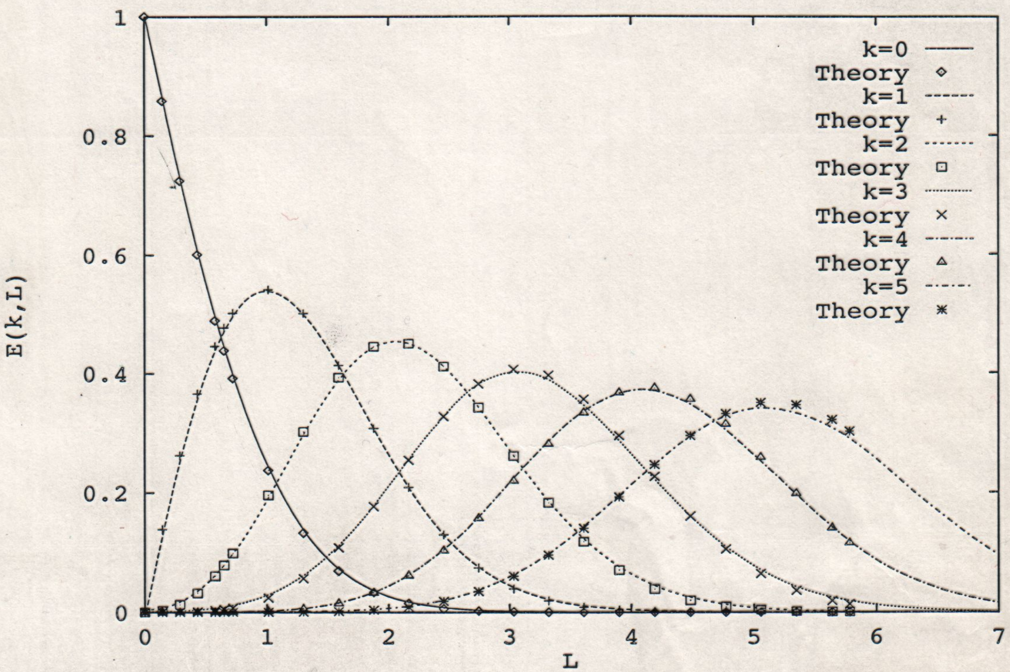


Figure 8: Same as in figure 1 but for 5168 consecutive levels of the quartic billiard (Prosen 1998) for $a = 0.04$ with sequential quantum number $\mathcal{N} \approx 8\,000\,000$, and for theoretical distributions with $\rho_1 = 0.12$.

quartic billiard $a = 0.04$

$r = 1 + a\cos(4\phi)$

Prosen and Robnik 1999

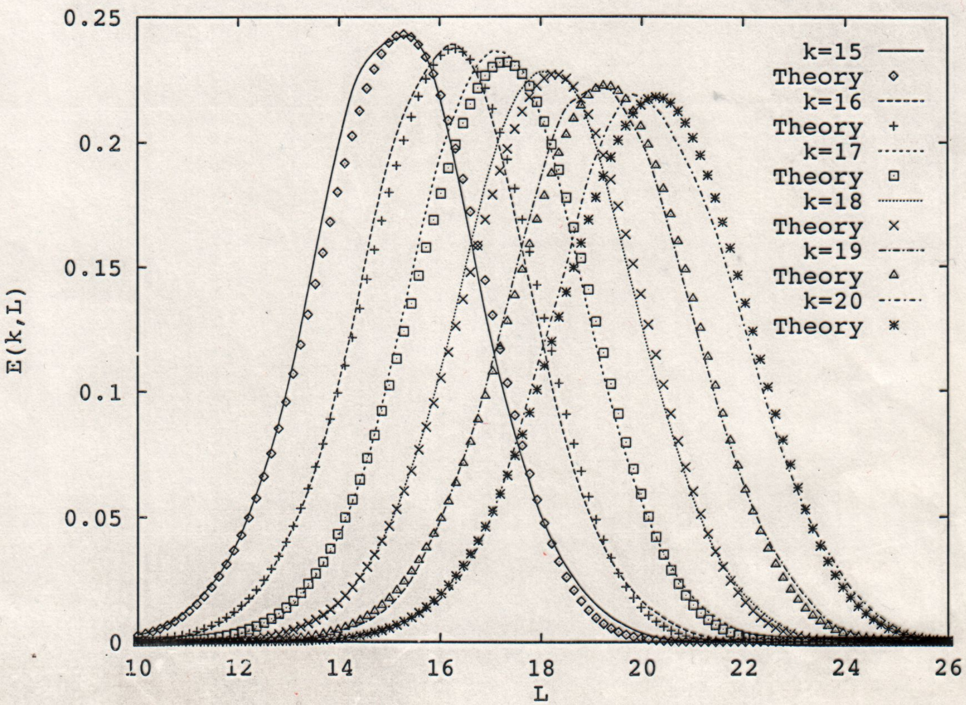


Figure 9: Same as in figure 8 but for $15 \leq k \leq 20$.

quartic billiard $a = 0.04$

$r = 1 + a\cos(4\phi)$

According to our theory, for a two-component system, $j=1,2$, we have (Berry-Robnik 1984):

Poisson (regular) component: $E_1(0,S) = e^{-S}$

Chaotic (irregular) component: $E_2(0,S) = \mathrm{Erfc}\left(\frac{\sqrt{\pi}S}{2}\right)$ (Wigner = 2D GOE)

$E(0,S) = E_1(0,\mu_1 S)E_2(0,\mu_2 S) = e^{-\mu_1 S}\mathrm{Erfc}(\frac{\sqrt{\pi}\mu_2 S}{2})$, where $\mu_1+\mu_2 = 1.$

Then $P(S)$ = level spacing distribution = $\frac{d^2E(0,S)}{dS^2}$ and we obtain:

$P_{BR}(S) = e^{-\mu_1 S}\left(\exp(-\frac{\pi\mu_2^2 S^2}{4})(2\mu_1\mu_2 + \frac{\pi\mu_2^3 S}{2}) + \mu_1^2\mathrm{Erfc}(\frac{\mu_2\sqrt{\pi}S}{2})\right)$
(Berry-Robnik 1984)

This is a one parameter family of distribution functions with normalized total probability $<1>=1$ and mean level spacing $<S>=1$, whilst the second moment can be expressed in the closed form:

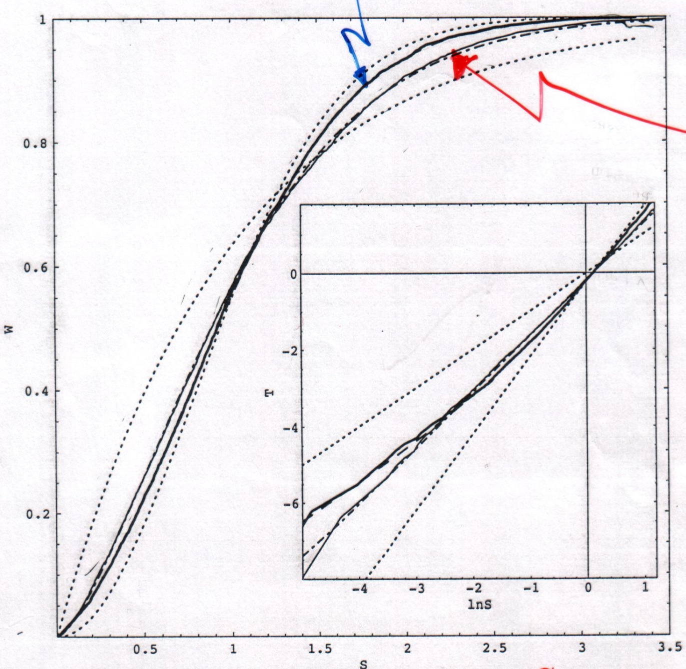
$<S^2>= 2\int_0^\infty E(S)\,dS = \frac{2}{\mu_1}\left(1 - e^{\frac{\mu_1^2}{\pi\mu_2^2}}\,\mathrm{Erfc}\left(\frac{\mu_1}{\sqrt{\pi}\mu_2}\right)\right)$ = 2 (Poisson), $4/\pi$ (GOE)

$\Delta \psi + k^2 \psi = 0, \quad \psi|_{\partial B} = 0$

T. Prosen 1998

Berry-Robnik (far semiclassics)
$k \approx 16000$, $\rho_1^q = 0.119$
$\rho_1^{cl} = 0.115 \pm 0.005$

$W(S) = \int_0^S P(x)dx$



near semiclassics
$k \approx 500$
Brody
$\beta = 0.46$

S

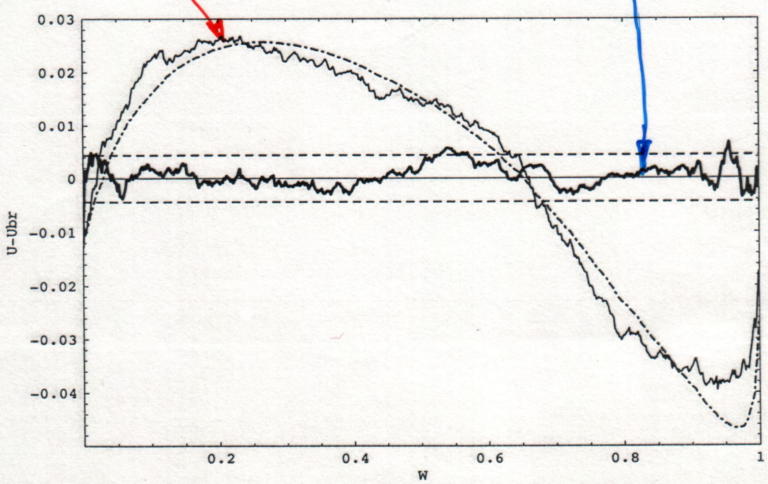
Figure 2: Cumulative nearest level spacing distribution $W(S)$ for a stretch of 5168 consecutive levels in the far semiclassical regime ($k \approx 16000$) (thick curve) and a stretch of 6220 consecutive levels in the near semiclassical regime ($k \approx 500$) (thin curve). The first numerical curve is almost overlapping with theoretical best fitting BR distribution for $\rho_1^q = 0.119$ (dashed curve), while the second numerical curve agrees very well with the best fitting Brody distribution with exponent $\beta = 0.46$ (dot-dashed curve). For comparison we give Poisson and GOE integrated level spacing distributions (dotted curves). In the inset we plot the same data in the T-function representation [21], $T(S) = \ln(-\ln(1 - W(S)))$ against $\ln S$, which transforms the Brody distributions (an hence also Poissonian and Wigner) to straight lines, and enhances the region of small spacings.

T. Prosen 1998

near semiclassics
Brody $k \approx 500$
$\beta = 0.46$

$k \approx 16000$, far sem.
Berry-Robnik
$\rho_1 = 0.119$

$U - U_{BR}$



W

Figure 3: Fine detail deviations from Berry-Robnik distribution (for $\rho_1 = 0.119$) in a uniform U-function transformation [21]: we plot $U(W(S)) - U(W_{\rm BR}(S))$ against $W(S)$. In the far semiclassical regime $k \approx 16000$ (5168 consecutive levels), the difference of U-functions (thick curve) lies within a band of expected statistical error δU (dashed lines), while in the near semiclassical regime $k \approx 500$ (6220 consecutive levels), the difference of U-functions (thin curve) agrees very well with the difference of U-functions for the best fitting Brody distribution with exponent $\beta = 0.46$ (dash-dotted curve).

Def.: $U = \frac{2}{\pi} \arccos \sqrt{1 - W(s)}$

4. New approach to describe the transition regime of spectral correlations

Let us consider an ensemble of real symmetric 2D matrices

$\begin{pmatrix} x & y \\ y & -x \end{pmatrix}$ with x, y real

the eigenvalue $\lambda = \pm\sqrt{x^2+y^2}$ and level spacing $S = \lambda_1 - \lambda_2 = 2\sqrt{x^2+y^2}$

The level spacing distribution is $P(S) = \int_{R^2} dx\, dy\, g_x(x) g_y(y) \delta(S - 2\sqrt{x^2+y^2})$

Now we introduce a statistical ensemble by choosing $g_x(x)$ and $g_y(y)$.

In particular, we choose such $g_x(x)$ that if $g_y(y) = \delta(y)$ (diagonal matrix) the level spacing distribution $P(S)$ is equal to our $P_{BR}(S)$ (Berry-Robnik 1984):

$$P_{BR}(S) = e^{-\mu_1 S}\left(\exp(-\tfrac{\pi\mu_2^2 S^2}{4})(2\mu_1\mu_2 + \tfrac{\pi\mu_2^3 S}{2}) + \mu_1^2 \mathrm{Erfc}(\tfrac{\mu_2\sqrt{\pi}S}{2})\right)$$

After a short calculation: $g_x(x) = P_{BR}(2x)$.

Introducing the polar coordinates (r, φ) instead of (x, y), we have

$$P(S) = \int_0^{2\pi} d\varphi \int_0^{\infty} r\, dr g_x(r\cos\varphi) g_y(r\sin\varphi)\delta(S-2r)$$

$$P(S) = \frac{S}{4}\int_0^{2\pi} d\varphi\ g_x\left(\frac{S}{2}\cos\varphi\right)\ g_y\left(\frac{S}{2}\sin\varphi\right).$$

Linear level repulsion is robust: $P(S) \approx \frac{\pi S}{2} g_x(0) g_y(0)$

Now we choose $g_x(x) = P_{BR}(2x)$ for the diagonal elements x

and Gaussian distribution for the offdiagonal elements such that σ will play the role of the perturbation or coupling parameter:

$$g_y(y) = \frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{y^2}{2\sigma^2})$$

and we get immediately (Stöckmann 2006, Vidmar et al 2007):

$$P_{DBR}^{A}(S) = \frac{S}{\sigma\sqrt{2\pi}}\int_0^{\pi/2} d\varphi\ P_{BR}\left(S\cos\varphi\right)\ \exp\left(-\frac{S^2\sin^2\varphi}{8\sigma^2}\right)$$

which is now a two-parameter family of level spacing distributions parametrized by the Berry-Robnik parameter μ_1 and the coupling parameter σ: **2D random matrix model for all-to-all couplings**

If instead only couplings between the regular and chaotic levels due to tunnelling are considered we must assume

$$g_y(y) = 2\mu_1(1-\mu_1)\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma^2}\right) + [1-2\mu_1(1-\mu_1)]\delta(y)$$

and obtain immediately:

$$P_{DBR}^{T}(S) = 2\mu_1(1-\mu_1)P_{DBR}^{A}(S) + [1-2\mu_1(1-\mu_1)]P_{BR}(S)$$

which is a **2D random matrix model for tunneling couplings between the regular and chaotic energy levels**

Limiting cases of $P^A_{DBR}(S)$:

$$P^A_{DBR}(S) = \frac{S}{\sigma\sqrt{2\pi}} \int_0^{\pi/2} d\varphi P_{BR}(S\cos\varphi) \exp\left(-\frac{S^2 \sin^2\varphi}{8\sigma^2}\right)$$

Small S: $P^A_{DBR}(S) = \frac{S\sqrt{\pi}}{2\sigma} P_{BR}(0)$

It has always a linear rise with the slope $\propto 1/\sigma$

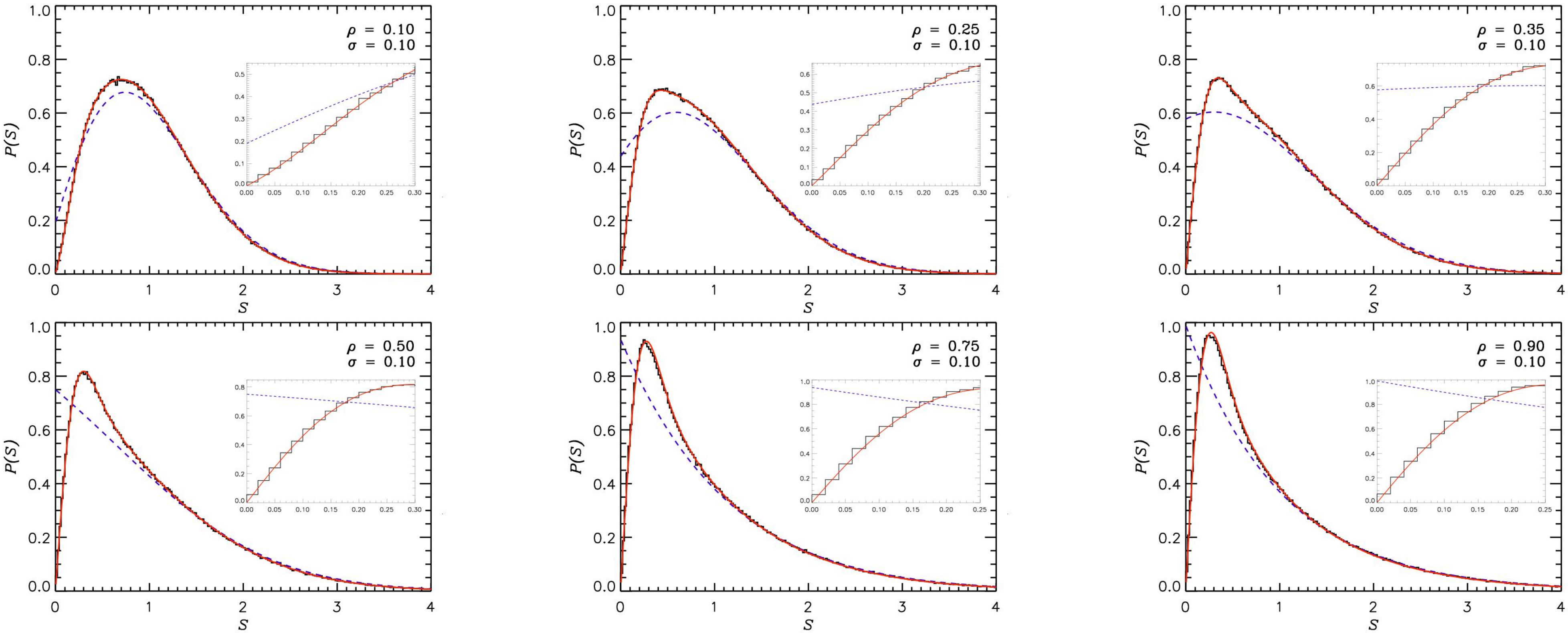
It can be improved by the power/series expansion of $P_{BR}(S) = \sum_{k=0}^{\infty} a_k S^k$.

Large S: expansion around $\varphi = 0$ to give large S asymptotics:

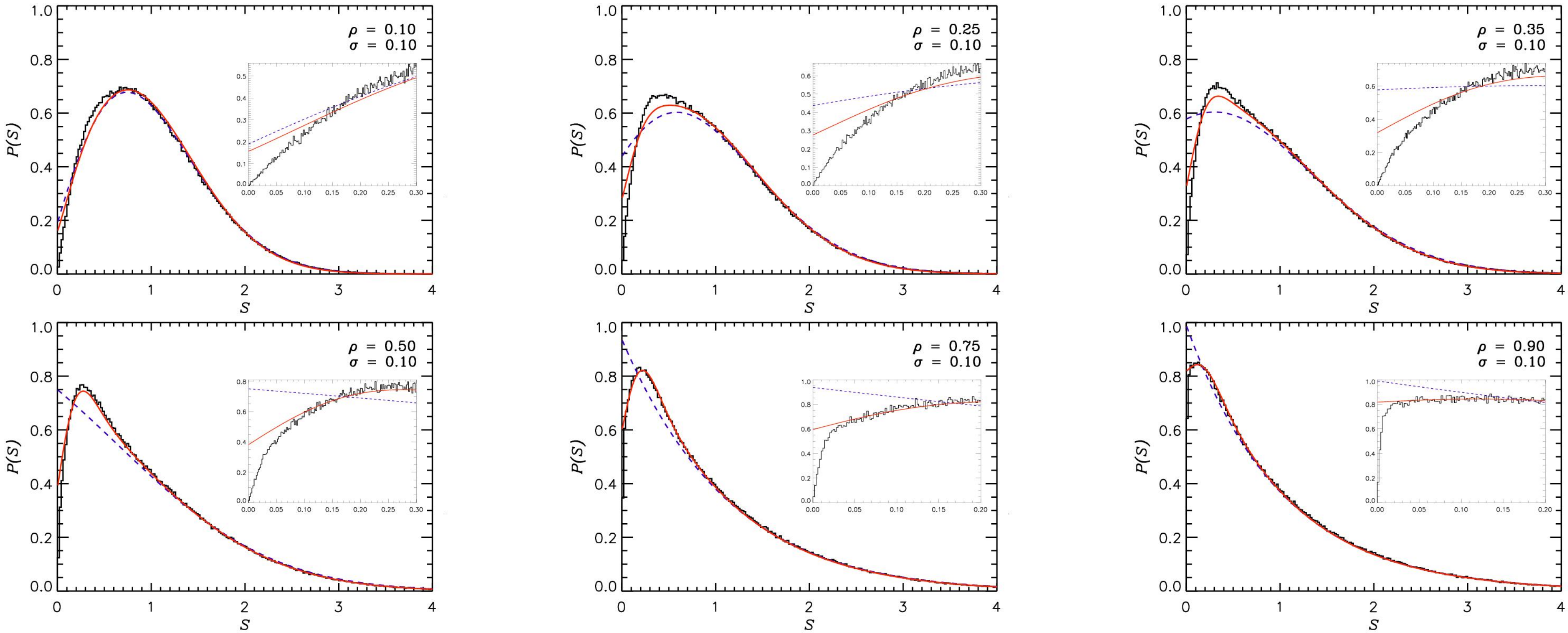
$$P^A_{DBR}(S) \approx \frac{S}{\sigma\sqrt{2\pi}} \int_0^{\infty} d\varphi P_{BR}(S) \exp\left(-\frac{S^2\varphi^2}{8\sigma^2}\right) = P_{BR}(S)$$

Can be improved by approximating $P_{BR}(S\cos\varphi) \approx P_{BR}(S) - \frac{1}{2}\varphi^2 S \frac{dP_{BR}}{dS}$ yielding:

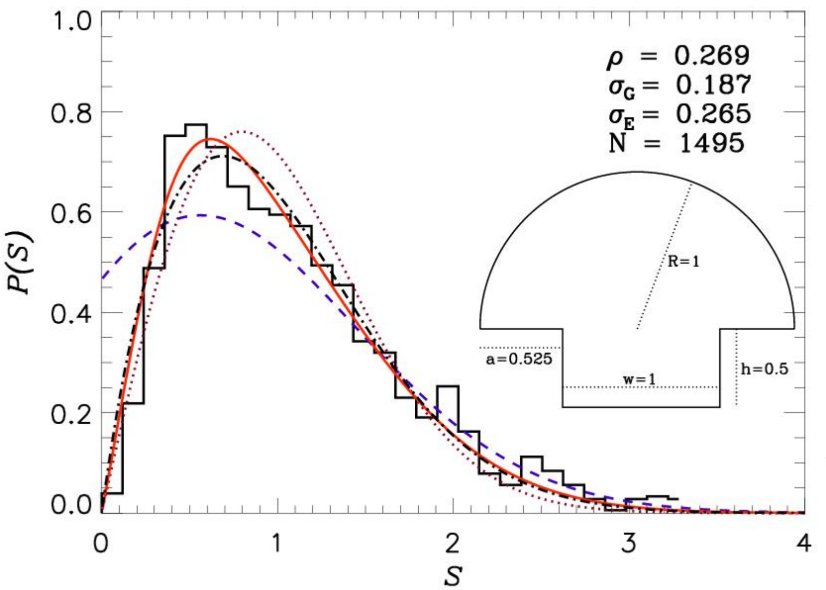
$$P^A_{DBR}(S) \approx P_{BR}(S) - \frac{2\sigma^2}{S} \frac{dP_{BR}(S)}{dS}$$

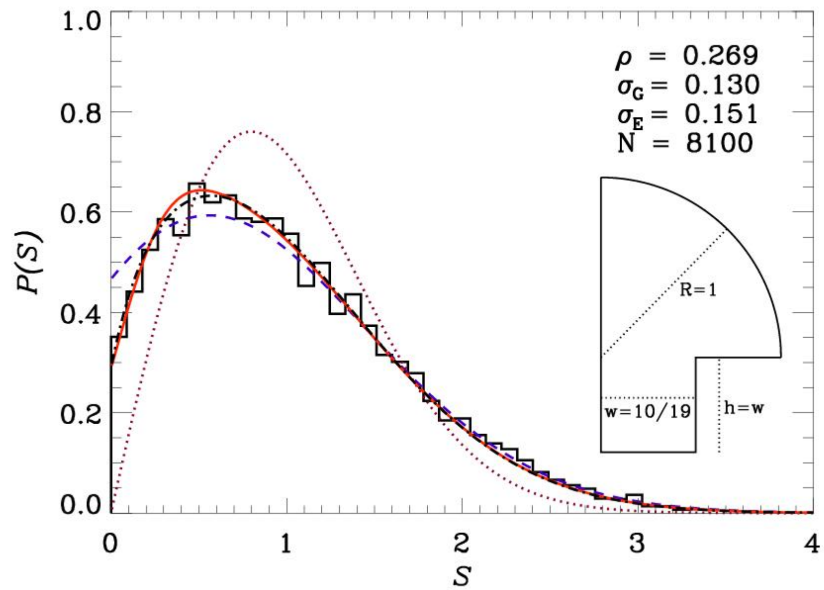


Antenna distorted BR distribution $P_{DBR}^{An}(S)$ (all-to-all level couplings)

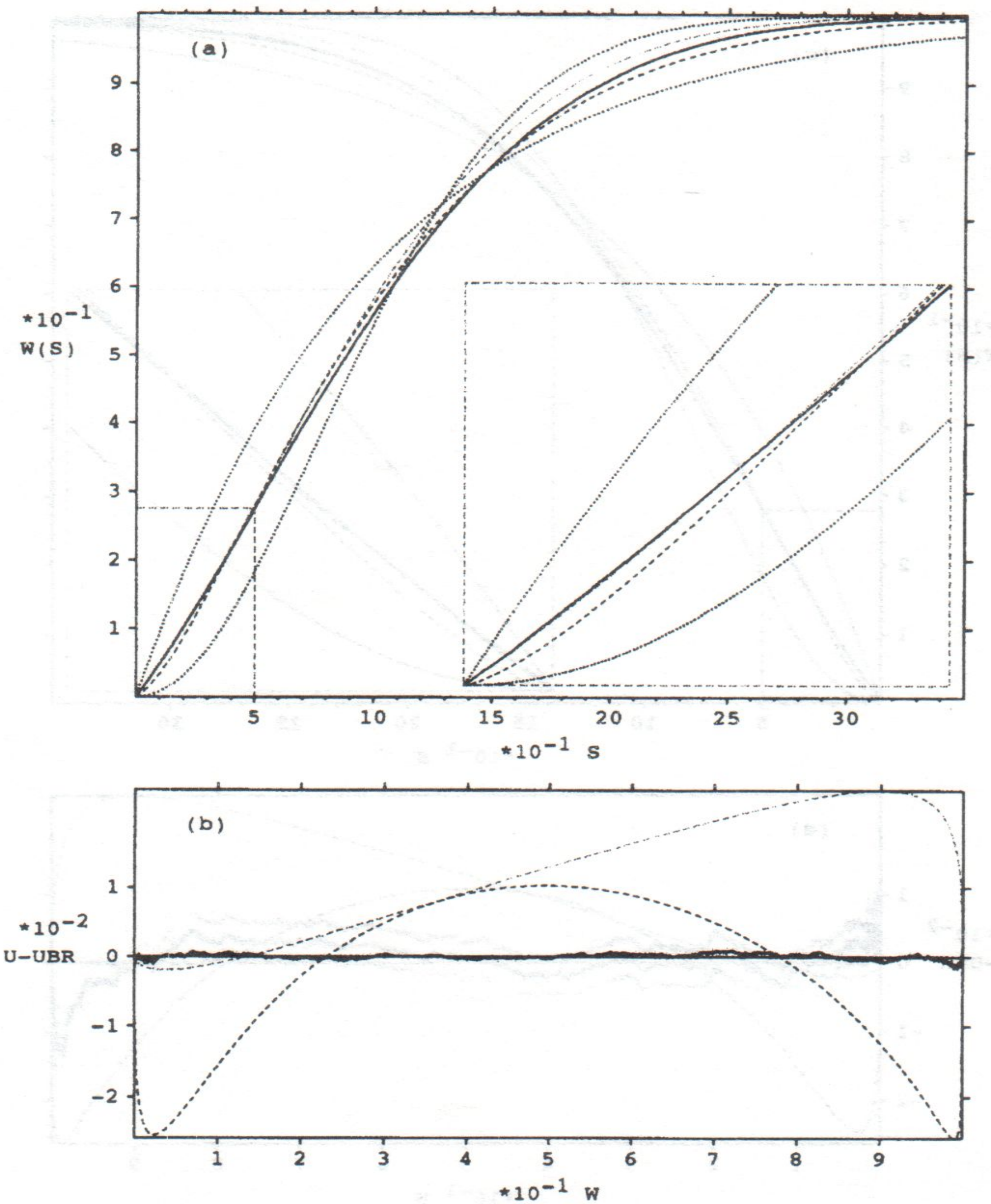


Tunnelling distorted BR distribution $P_{DBR}^{Tn}(S)$: only (tunneling) couplings between the regular and chaotic levels are allowed.





Left: Comparison of experimental data (histogram) with the best fitting theoretical curves for P_{DBRN}^{An}: Full line for the Gaussian model, dash-dotted for the exponential model, dashed for BR (with the same ρ), and dotted for the Wigner distribution.
Right: Comparison of the numerical data (histogram) with the best fitting theoretical curves for P_{DBRN}^{Tn}: Full line for the Gaussian model, dashed-dotted for the exponential model, dashed for BR (with the same ρ), and dotted for the Wigner distribution. σ_G and σ_E are the best fitting values of σ for the Gaussian and the exponential model, respectively. N_o is the number of objects in the histrogram. For other details see text.


(a)
*10^-1
W(S)
*10^-1 S
(b)
*10^-2
U-UBR
*10^-1 W

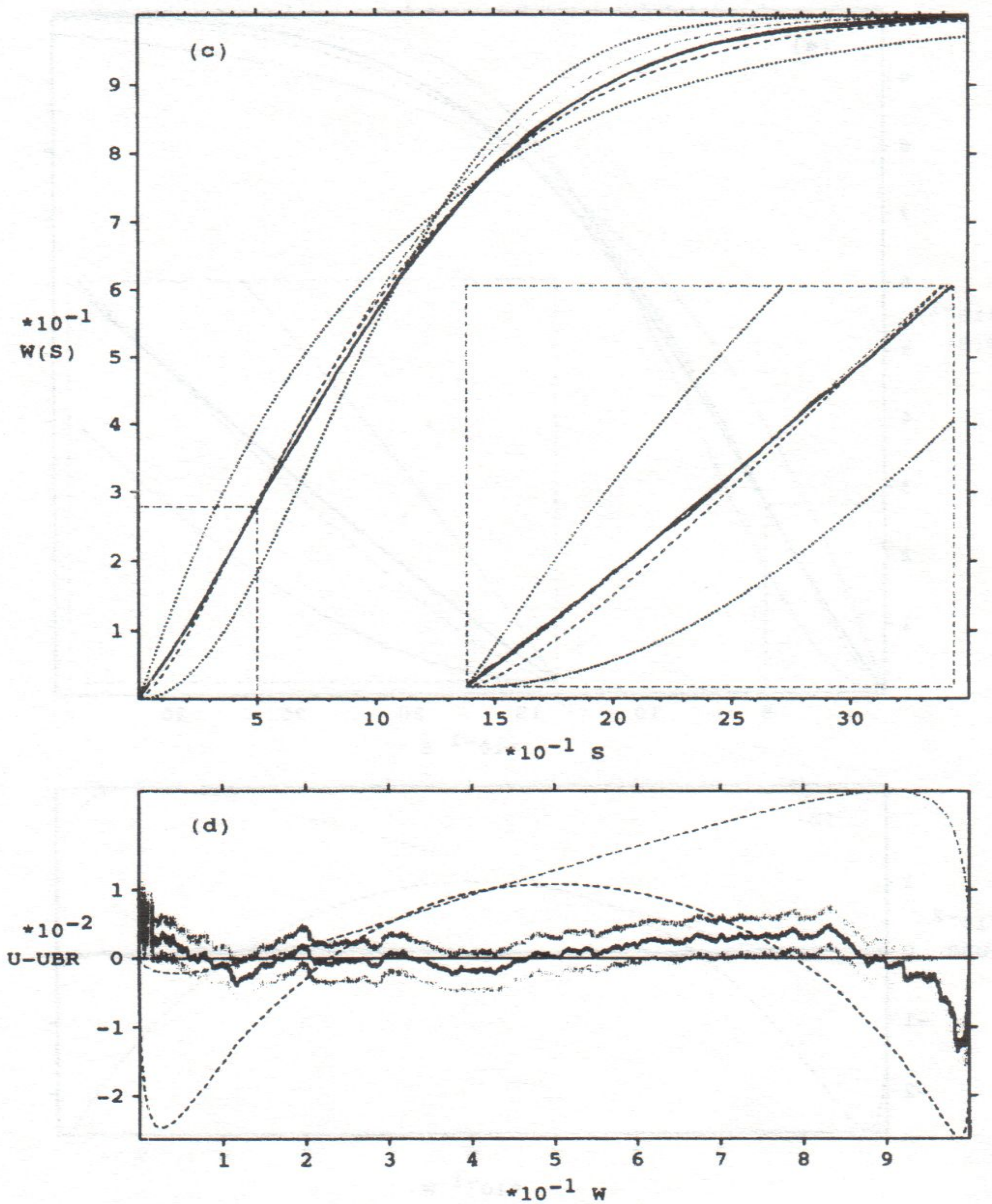

(c)
*10^-1
W(S)
*10^-1 s
(d)
*10^-2
U-UBR
*10^-1 W

Figure 1. (a), (c) show the results for $W(S)$ and (b)–(d) show the so-called U-function $U(W) - U(W_{\mathrm{BR}})$. Here W_{BR} refers to the best fitting Berry–Robnik level spacing distribution, so that abscissa in the diagrams (b), (d) is the ideal agreement with Berry–Robnik statistics. The results for the quantized compactified standard map are in (a), (b) and for the two-dimensional semiseparable autonomous Hamiltonian harmonic oscillator in (c), (d). The full heavy curve is data, the full light curve is the best-fitting Berry–Robnik, the broken curve is best-fitting Brody and the chain curve is the best-fitting Abul-Magd. Abul-Magd is the upper curve and Brody is the lower one. It is clearly seen for big S in the W-plots that the disagreement with Abul-Magd's prediction is very bad on this global scale, and this disagreement turns out to be indeed very big in the U-function plots, except perhaps at small S. For the reference we plot here also the $\pm\sigma$ bands (grey) of expected statistical standard deviation. For the sake of completeness we quote the best fitting parameter values: In (a), (b) we have the classical $\rho_1 = 0.265$, the quantal Berry–Robnik $\rho_1 = 0.273$ and the quantal Abul-Magd $q = 0.448$. In (c), (d) we have the classical $\rho_1 = 0.291$, the quantal Berry–Robnik $\rho_1 = 0.286$ and the quantal Abul-Magd $q = 0.466$. In (a), (b) we have 160 000 numerical quasi-energy levels for quantum maps with dimensions 15 982–16 000, with the same kick parameter $a = 1.8$ and the same classical limit. In (c), (d) we have a stretch of 13 445 energy levels starting from around 17 684 000th level. In plots (a) and (c) we show for comparison also the GOE and Poissonian curves (dotted), and in the inset the magnification of the situation at small spacings S. In (a) the differences between the data and theory (Berry–Robnik) are not visible, whilst in (c) they can be seen, especially in the inset, whilst in both (b) and (d) the (quite small) differences between the data and theory (Berry–Robnik) are made visible.

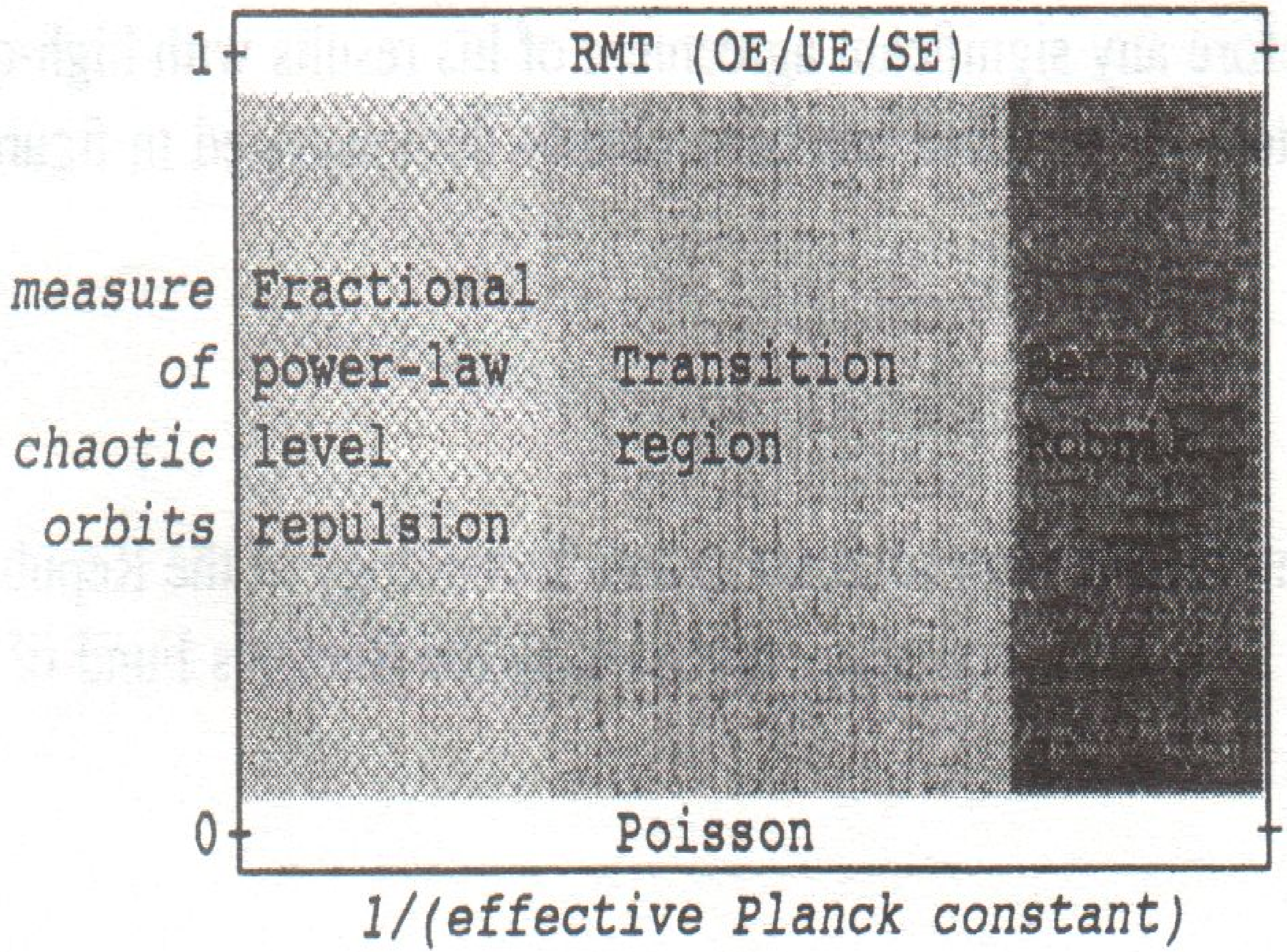


Figure 2. We show the schematic diagram of the doubly transition region: from integrable to ergodic classical dynamics and from near semiclassics (not very small $\hbar$) to far semiclassics (sufficiently small $\hbar$).

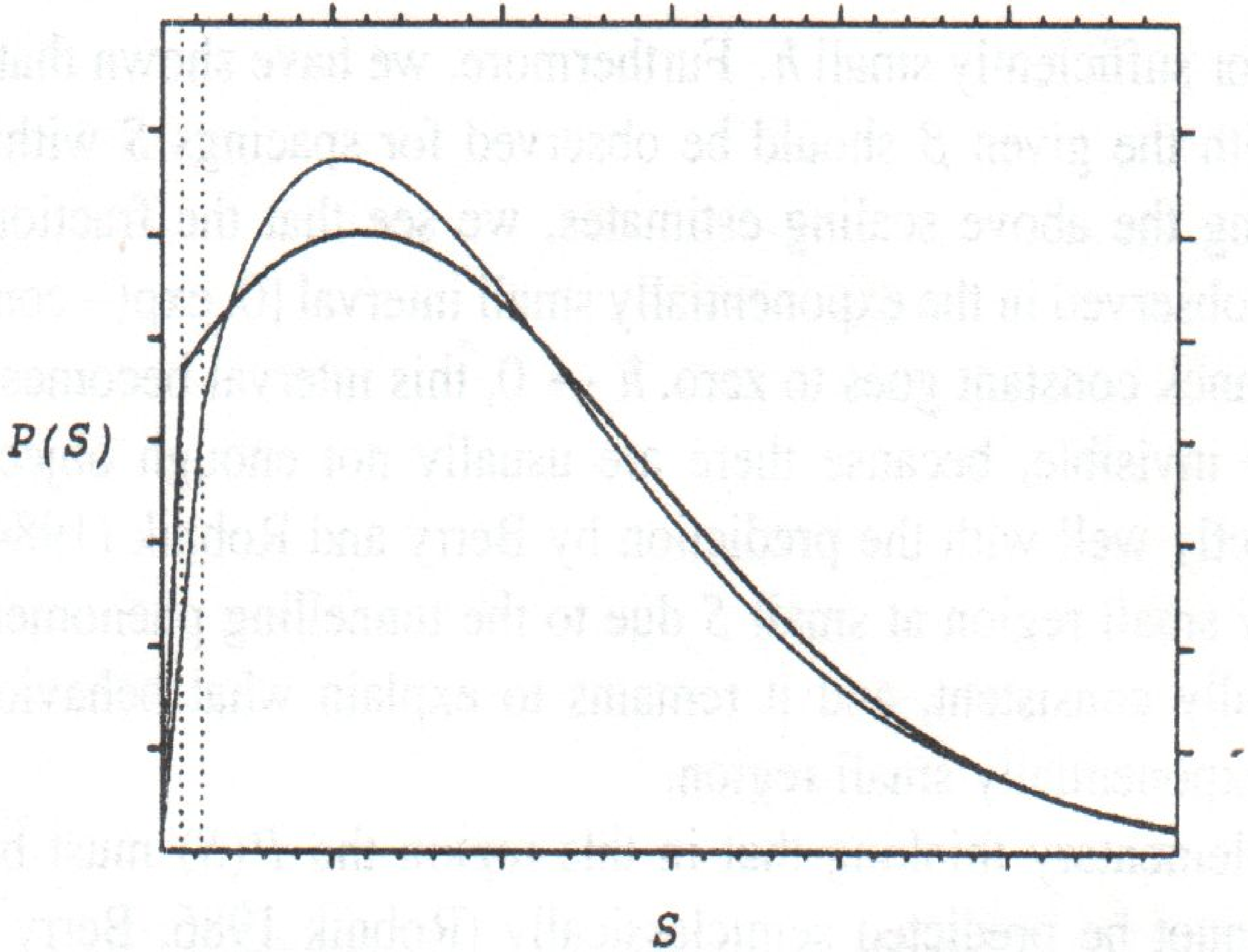


Figure 3. We show schematically two examples of the Brody-like level spacing distribution (with higher maximum) and Berry–Robnik type, but in both cases indicated the exponentially small (but here exaggerated) regime of linear level repulsion (see text).

Discussion and conclusions

• The Principle of Uniform Semiclassical Condensation of Wigner functions of eigenstates leads to the idea that in the sufficiently deep semiclassical limit the spectrum of a mixed type system can be described as a statistically independent superposition of regular and chaotic level sequences.

• As a result of that the $E(k, L)$ probabilities factorize and the level spacings, sigma and delta statistics can be calculated in a closed form.

• At low energies in the near semiclassical limit where the effective Planck constant is not sufficiently small, we see deviations from the uniform condensation (of WF), localization phenomena and tunneling between regular and chaotic levels as well as between regular and regular levels, and also between localized chaotic and chaotic levels.

• We propose a new 2-parameter family of level spacing distributions in terms of a 2D random matrix model (Stöckmann 2006, Vidmar et al 2007). Regular-regular correlations through the intermediary of a chaotic level as a second order effect (chaos assisted tunnelling) must be included to improve our results on $P^T_{DBR}(S)$.

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