Some Infinite Families of Complex Contracting Mappings

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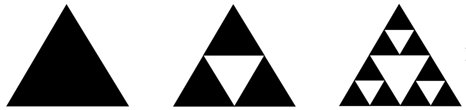
Selfsimilar (fractal) set

The idea of self-similarity is one of the most fundamental in modern mathematics and physics.

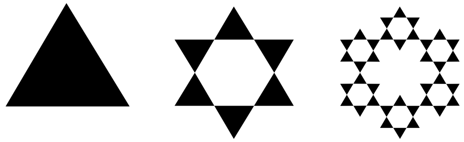
The notion of renormalization group, which plays an essential role in quantum field theory, statistical physics and dynamical systems, is related to it.

Many fractal and multi-fractal objects, playing an important role in singular geometry, measure theory and holomorphic dynamics, are also related.

A compact set F in a metric space is called self-similar (in a strong sense) with a similarity coefficient $r > 1$, when it can be divided into n congruent sets, each of which is exactly r-times smaller copy of the original set. The fractal dimension of such object d_F matches its self-similar dimension $D = d_S = \dfrac{\ln n}{\ln r}$.



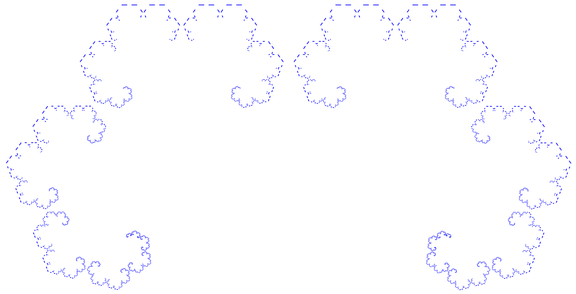
$$D = \frac{\ln 3}{\ln 2} \doteq 1{,}5850$$



$$D = \frac{\ln 6}{\ln 3} \doteq 1{,}631$$

Fractal sets generated by (symmetric) bushes

We have studied special families of strongly selfsimilar subsets of the complex plane, which can be obtained, on one hand, as "leaves" of planar trees and, on the other hand, as invariant sets (limits) of certain infinite families of contracting mappings - **I**terated **F**unction **S**ystems (homotheties with $q = \frac{1}{r} < 1$).

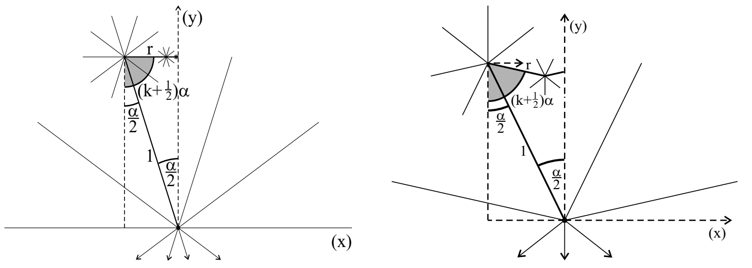


For any integer $n > 1$ we start at the origin (point 0 in the complex plane) with n symmetrically placed unit intervals, so that the angle between each two of these intervals equals $\alpha \leq \frac{2\pi}{n}$. At the end of every considered unit interval we place a smaller copy of the whole initial configuration, stretched by a positive factor $q < 1$.

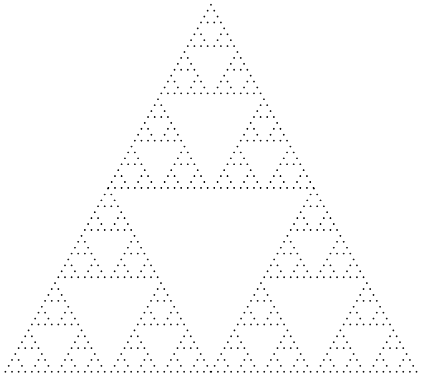
When there is no overlapping between the different branches of the constructed bush, the set of all endpoints of different paths in the obtained bush $F = F(n, r, \alpha)$ will obviously (by our construction) be selfsimilar in a strong sense, with the stretching factor q and the dimension $d_S = \frac{\ln n}{\ln q^{-1}}$.

At every vertex another parameter arrises, the minimal angle φ between the new (stretched) intervals and the original (unit) interval. We will demonstrate in details just two main (bordering) cases:

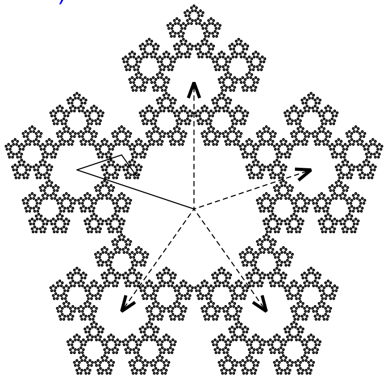
I. $\varphi = 0$, so the new interval is a continuation of the original one;
II. $\varphi = \frac{\alpha}{2}$ and the change of direction from the previous interval to its nearest successor is maximal.



When $n = 3$ the first construction gives also the well known Sierpinski gasket, while for $n = 4$ the resulting set entirely fills a square with side $2\sqrt{2}$ (so it is not a fractal!).



$n = 3,\ r = \frac{1}{2}$



$n = 5,\ r = \frac{3-\sqrt{5}}{2}$

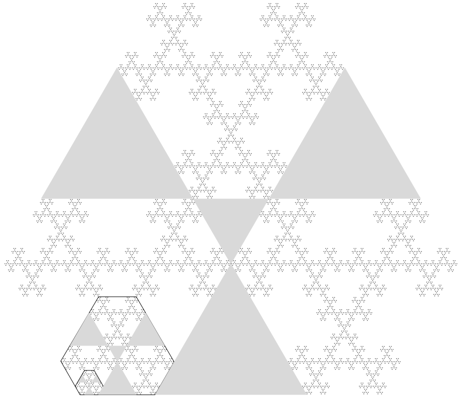
Theorem 1. *For $\alpha = \frac{2\pi}{n}$ and an arbitrary integer $n > 1$ the set $F(n, r, 0)$ is without overlapping exactly when*

$$r \leq \begin{cases} \dfrac{\sin\frac{\pi}{n}}{\sin[\frac{n+1}{4}]\frac{2\pi}{n}+\sin\frac{\pi}{n}}, & \text{if } n \text{ is odd} \\ \dfrac{\sin\frac{\pi}{n}}{\sin(2[\frac{n+1}{4}]+1)\frac{\pi}{n}+\sin\frac{\pi}{n}}, & \text{when } n \text{ is even.} \end{cases}$$

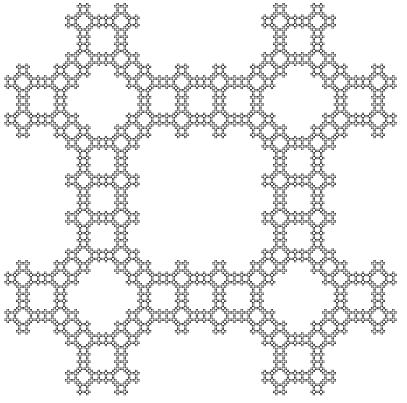
In the following table the values of the angle α (in degrees), maximal stretching factor r, for which there is no overlapping, and the dimension d_S of the obtained symmetrical (fractal) set $F(n, r, 0)$ are listed for the first few integers n.

n	2	3	4	5	6	7	8	9	10	11	12
α	180	120	90	72	60	51,43	45	40	36	32,73	30
r	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0,382	$\frac{1}{3}$	0,308	0,293	0,258	0,236	0,222	0,211
d_S	1	1,585	2	1,672	1,63	1,652	1,693	1,621	1,595	1,591	1,599

Similar results are obtained in case II and the resultings fractal sets are not much different.



$n = 3, \; r = \frac{1}{2}, \; \varphi = 60^\circ$



$n = 4, \; r = \frac{\sqrt{5}-1}{\sqrt{2}}, \; \varphi = 45^\circ$

Theorem 2. *For every $n > 1$ there is no overlapping in the set $F(n, r, \frac{\pi}{n})$ exactly when*

$$r \leq \begin{cases} \dfrac{\sin\frac{\pi}{n}}{\sin[\frac{n+2}{4}]\frac{2\pi}{n} + \sin\frac{\pi}{n}}, & \text{if } n \text{ is odd} \\ \dfrac{\sqrt{\sin^2[\frac{n}{4}]\frac{2\pi}{n} + 8\sin\frac{\pi}{n}\sin([\frac{n}{4}]+1)\frac{\pi}{n}\cos[\frac{n}{4}]\frac{2\pi}{n}} - \sin[\frac{n}{4}]\frac{2\pi}{n}}{4\sin([\frac{n}{4}]+1)\frac{\pi}{n}\cos[\frac{n}{4}]\frac{2\pi}{n}}, & \text{for even } n. \end{cases}$$

Below we list some values of the maximal stretching factor r and the dimension d_S of the obtained fractal set $F(n, r, \frac{\pi}{n})$:

n	2	3	4	5	6	7	8	9	10	11	12
r	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$0,437$	$0,382$	$0,357$	$0,308$	$0,280$	$0,258$	0,243	0,222	0,207
d_S	2	$1,585$	$1,675$	$1,672$	$1,740$	$1,652$	$1,634$	1,622	1,628	1,593	1,578

Fractals generated by infinite IFS

Another way of generating tree fractal sets (from a single interval) is based on iterations of a family of affine mappings.

In a complete metric space (X, d) a family of contractions $\mathcal{S} = \{f_k,\ k = 1, 2, \ldots\}$ with scaling factors $q_k < 1$ is called a hyperbolic **system of iterated functions (IFS)**. The following classic statement describes the main property of such systems:

Theorem. For every IFS and an arbitrary element B of $\mathcal{H}(X)$ - the space of all nonempty compact subsets of X with the Hausdorff metrics, there exists the limiting set $\widetilde{B} = \lim_{j\to\infty} \mathcal{S}^j(B)$, where $\mathcal{S}^j(B) \stackrel{\text{def}}{=} \bigcup_k f_k(\mathcal{S}^{j-1}(B))$, which is a fixed point (**actractor**) of the given IFS.

For an arbitrary stretching factor $r \in (0,1)$ and a positive angle $\alpha < \pi$ we define an infinite family of bijective affine mappings of the complex plane $\{\varphi_k\}, k > 0$:

$$\varphi_{2k-1}(z) = \varphi_{r,\alpha,2k-1}(z) = r^{k+1} e^{i(k-1)\alpha}(z - \tfrac{1}{2}) + \sum_{j=0}^{k-1} r^j e^{ij\alpha} - \tfrac{1}{2}$$

$$\varphi_{2k}(z) = \varphi_{r,\alpha,2k}(z) = r^{k+1} e^{-i(k-1)\alpha}(z + \tfrac{1}{2}) - \sum_{j=0}^{k-1} r^j e^{-ij\alpha} + \tfrac{1}{2}.$$

So, each of these mappings is a composition of a rotation of the complex plane (for a certain multiple of the angle α) with the appropriate stretching (for the related power of factor r) and some translation. Also $\varphi_{2k}(-\overline{z}) = -\overline{\varphi_{2k-1}(z)}$ for every complex number z, yielding that images $\varphi_{2k-1}(z), \varphi_{2k}(-\overline{z})$ of any two complex points $z, -\overline{z}$, symmetric in relation to the imaginary axis, are again symmetric towards this axes.

Finally, choosing a closed unit interval $F_0 = [-\frac{1}{2}, \frac{1}{2}]$ on the real axis as a starting set, we define the sequence $\{F_j\}, j \geq 0$ of subsets of the complex plane as the subsequent images obtained from F_0 by iteration of all mappings from the given family:

$$S_j = \bigcup_{k>0} \varphi_k(S_{j-1}).$$

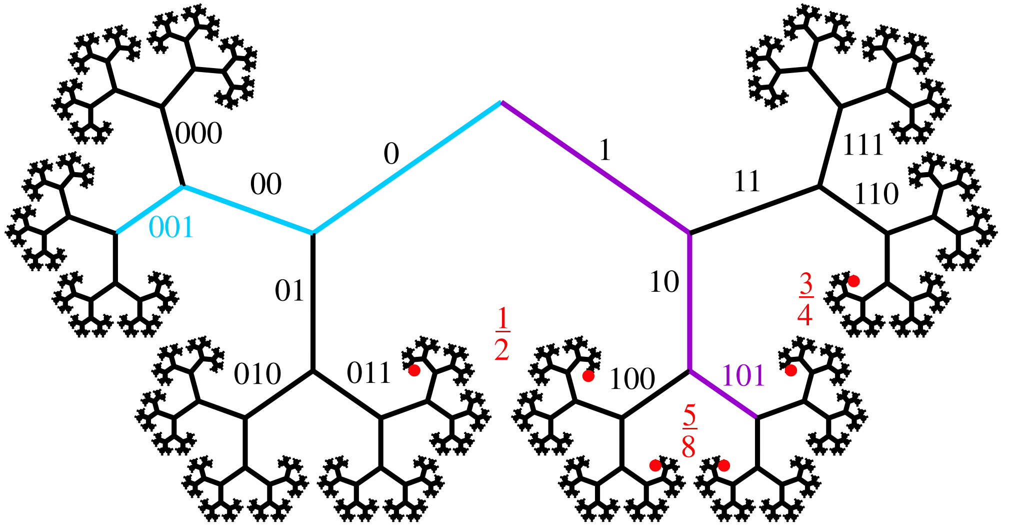
Theorem.The limiting set $S_{r,\alpha} = \lim_{j\to\infty} S_j$ is a tree fractal $F(2, r, \alpha)$.

Encodings of elements of a (unit) interval

A standard way to write down the real numbers from the [0, 1] interval, relative to an arbitrary integer base $b > 1$, is:

$$x = 0, t_1 t_2 t_3 \ldots_b = \frac{t_1}{b} + \frac{t_2}{b^2} + \frac{t_3}{b^3} + \ldots, \text{ where all } t_j \in \{0, 1, 2, \ldots, b-1\}.$$

We have a 1-1 correspondence between the unit interval and the (1-dimensional) tree fractal bordering line, in repect to which the decimal numbers in notation of any nonnegative real number $x \leq 1$ (relating to base b) indicate the path through these tree from its origin to the considered limiting point. This model is evidently linear.


000
00
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10
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011
100
101
5/8

Continuing fractions

Another way to present a number on the interval (0,1) is a (regular) **continuing fraction** $[n_1, n_2, n_3, \ldots] = \dfrac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \ldots}}}$.

Here every irrational number is connected to a unique infinite sequence of natural numbers, while finite sequences of positive numbers correspond to rational numbers on the unit interval:

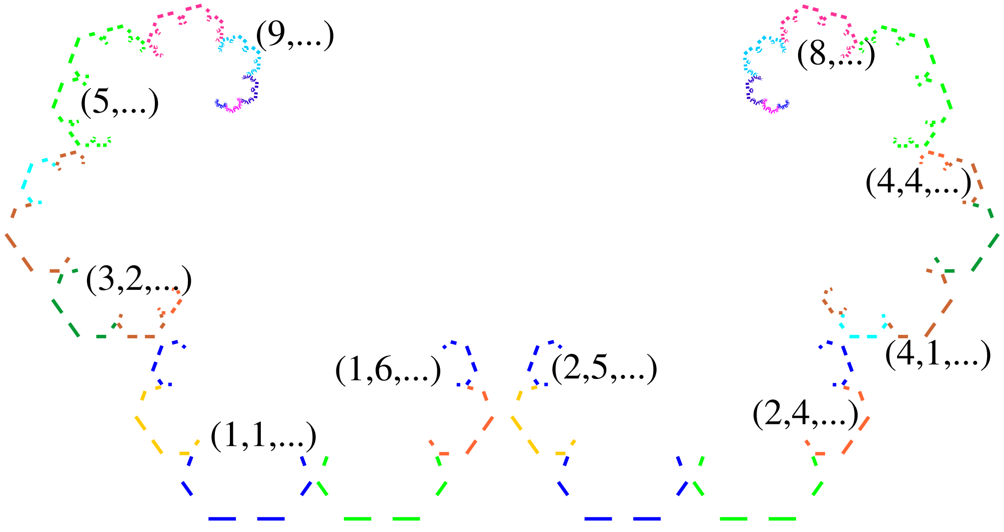
$$\begin{array}{ll} \frac{21}{32} = [1,1,1,10] & 2-\sqrt{3} = [3,1,2,1,2,1,2,1,2,1,\ldots] \\ \frac{22}{32} = [1,2,5] & \frac{\sqrt{5}-1}{2} = [1,1,1,1,1,\ldots] \\ \frac{23}{32} = [1,2,1,1,4] & \frac{1}{\sqrt[3]{2}} = [1,3,1,5,1,1,4,1,1,8,1,\ldots] \\ \frac{71}{263} = [3,1,2,2,1,1,1,2] & \pi - 3 = [7,15,1,292,1,1,1,2,1,3,\ldots] \end{array}$$

As each component (point) of the limiting tree fractal set $F(n, r, \alpha)$ is the result of application of a unique sequence of our infinitely many contracting mappings $f_k^{r,\alpha}, \ k > 0$

$$\{\ldots, f_{n_3}, f_{n_2}, f_{n_1}\}$$

to the original interval, we obtain another way of representing real numbers from the interval [0,1] by sequences of natural numbers:

$$x \mapsto (n_1, n_2, n_3, \ldots) \Longleftrightarrow x = f_{n_1}(f_{n_2}(f_{n_3}(\ldots([0,1])\ldots))).$$


(9,...)
(8,...)
(5,...)
(4,4,...)
(3,2,...)
(4,1,...)
(1,6,...)
(2,5,...)
(2,4,...)
(1,1,...)

Some examples - approximations of real numbers

Fraction	Binary code	Continuing fraction	IFS_2 induced code
$\frac{25}{256}$	00011001	[10,4,6]	[5,2,1]
$\frac{27}{128}$	00110110	[4,1,2,1,6]	[3,2,4]
$\frac{71}{256}$	01000111	[3, 1, 1, 1, 1, 6, 2]	[1,5,>4]
$\frac{11}{32}$	01011000	[2,1,10]	[1,1,2,>3]
$\frac{27}{64}$	01101100	[2, 2, 1, 2, 3]	[1,2,4,1]
$\frac{61}{128}$	01111010	[2,10,6]	[1,6,2]
$\frac{133}{256}$	10000101	[1, 1, 12, 3, 3]	[2,5,1]
$\frac{75}{128}$	10010110	[1, 1, 2, 2, 2, 4]	[2,1,1,2]
$\frac{157}{256}$	10011101	[1, 1, 1, 1, 2, 2, 2, 3]	[2,1,4,2]
$\frac{5}{8}$	10100000	[1,1,1,2]	[2,2,>7]
$\frac{95}{128}$	10111110	[1, 2, 1, 7, 4]	[2,10]
$\frac{209}{256}$	11010001	[1, 1, 1, 1, 2, 2, 2, 3]	[4,2,3]
$\frac{117}{128}$	11101010	[1, 10, 1, 1, 1, 3]	[6,2,2]
$\frac{249}{256}$	11111001	[1, 35, 1, 1, 3]	[10,1]