Magnetic domain patterns under an oscillating fields

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Domain Patterns

A wide variety of physical and chemical systems display domain patterns: for example,

- Thermal convection in fluids
- Chemical reaction systems
- Ferromagnetic thin films
  Ferrofluids
- Superconductors
- Biological media
  etc.
Magnetic Domain Patterns

Let us consider a ferromagnetic thin film like the schematic picture.

- It has strong uniaxial magnetic anisotropy.
- Its easy axis is perpendicular to the film.
- Because of interactions between spins, up and down spins form clusters (domains).
Outline

1. Model and Method
   for numerical simulations

2. Labyrinth $\rightarrow$ Stripes $\rightarrow$ Lattice
   typical domain patterns under an oscillating field

3. Traveling pattern
   equation for slow motion

4. Concentric circles, Spiral pattern
   some interesting patterns

5. Summary
Model & Equation

Simple two-dimensional Ising-like model. The Hamiltonian consists of 4 terms written by using a scalar field $\phi(r)$.

1. **uni-axial anisotropy:**

$$H_{\text{ani}} = \alpha \int \text{d}r \left( -\frac{\phi(r)^2}{2} + \frac{\phi(r)^4}{4} \right)$$

2. **external field:**

$$H_{\text{ex}} = -h(t) \int \text{d}r \phi(r)$$
3. exchange interactions:

\[ H_J = \beta \int d\mathbf{r} \frac{|\nabla \phi(\mathbf{r})|^2}{2} \]

4. dipolar interactions:

\[ H_{di} = \gamma \int d\mathbf{r} d\mathbf{r}' \phi(\mathbf{r}) \phi(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \]

\[ G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \text{ at long distances.} \]

Then the dynamical equation is described by

\[ \frac{\partial \phi(\mathbf{r})}{\partial t} = -\delta \frac{H_{ani} + H_J + H_{di} + H_{ex}}{\delta \phi(\mathbf{r})} \]
The equation in Fourier space

\[ \frac{\partial \phi_k}{\partial t} = \underbrace{(\alpha - \beta k^2 - \gamma G_k)}_{\eta_k} \phi_k + h(t) \delta_{k,0} - \phi^3 |_k \]

Here, \( \cdot |_k \) means the convolution sum, and

\[ G_k = a_0 - a_1 k, \quad (k = |k|) \]

\[ a_0 = 2\pi \int_{d}^{\infty} r dr G(r) = 2\pi / d, \quad a_1 = 2\pi \]

\( d \): cutoff length, which is fixed as \( d = \pi / 2 \) below.
Linear Growth Rate

Let us consider only linear terms in the equation:

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\phi_k \text{ decays for } \eta_k < 0 \\
\phi_k \text{ grows for } \eta_k > 0 
\end{cases}
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(But the nonlinear term prevents \( \phi_k \)'s growing too much.)
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\[
\eta_k = -(\beta k^2 - \gamma a_1 k + \gamma a_0) + \alpha \\
= -\beta \left( k - \frac{a_1 \gamma}{2\beta} \right)^2 + \frac{a_1^2 \gamma^2}{4\beta} - \gamma a_0 + \alpha
\]

The characteristic length of domain patterns should be \( 2\pi/k_0 \).

Here, we set \( \beta = 2.0, \gamma = 2/\pi \Rightarrow k_0 = 1 \).
Experiments

Examples of experimentally observed domain patterns under oscillating fields

- The labyrinth structure changes into parallel-stripes when the field is not very strong.

- When the field amplitude is increased, a lattice structure appears.

[Courtesy of Prof. Mino (Okayama Univ.): Experiments in iron garnet films.]
Numerical Simulations

External field: \[ h(t) = h_0 \sin \omega t; \quad \omega = 2\pi \times 10^{-2} \]

- \( h_0 \) is not large; \( h_0 = 0.72. \) (\( \alpha = 2.0 \))

- \( h_0 \) is large; \( h_0 = 1.15. \) (\( \alpha = 2.0 \))
ω-dependence of Lattice Formation

The lattice structure depends on the frequency ω.

ω = 2π × 2 × 10^{-2} \quad (\alpha = 2.0, \ h_0 = 1.15)

ω = 2π × 5 × 10^{-2} \quad (\alpha = 2.0, \ h_0 = 1.15)
Traveling Pattern

The whole pattern moves much more slowly than the field frequency.

$$\alpha = 2.0,$$

$$\omega = 2\pi \times 5 \times 10^{-2}$$

Ex. 1: $$h_0 = 0.80$$

Ex. 2: $$h_0 = 0.95$$
Traveling Pattern

The whole pattern moves much more slowly than the field frequency.

Basic mechanism: drift bifurcation (parity-breaking bifurcation) \([1,2]\) — a periodic pattern begins to drift when its second spatial harmonic is not damped strongly \((k-2k\) interaction).

\[
\alpha = 2.0, \\
\omega = 2\pi \times 5 \times 10^{-2}
\]

Ex. 1: \(h_0 = 0.80\)
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Dynamical Equation for Slow Motion

The patterns travel very slowly compared with the time scale of the field frequency.

How shall we analyze the traveling pattern theoretically?
Dynamical Equation for Slow Motion

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How shall we analyze the traveling pattern theoretically?

The dynamics under a rapidly oscillating field can be separated into a rapidly oscillating part and a slowly varying part.

— Kapitza’s inverted pendulum [3]

Kapitza’s Inverted Pendulum

When a rapidly oscillating force is applied to a pendulum, the unstable stationary point can turn to a stable point.
Kapitza’s Inverted Pendulum

When a rapidly oscillating force is applied to a pendulum, the unstable stationary point can turn to a stable point.

The equation of motion is

\[ m\ddot{x} = -\frac{dU}{dx} + f. \]

\(f\): a force oscillating rapidly (frequency: \(\omega\)).

Let us separate \(x(t)\) into a slowly varying part \(X(t) = \bar{x}\) and a small rapidly oscillating part \(\xi(t)\):

\[ x(t) = X(t) + \xi(t). \]
Effective Potential

Expanding in powers of $\xi$ as far as the first order terms, we obtain

$$m\ddot{X} + m\dot{\xi} = -\frac{dU}{dx} - \xi \frac{d^2U}{dx^2} + f(X, t) + \xi \frac{\partial f}{\partial X}. \quad (*)$$

For the oscillating terms,

$$m\ddot{\xi} = f(X, t) \quad \rightarrow \quad \xi = -\frac{f}{m\omega^2}$$

We average Eq. (*) with respect to time:

$$m\ddot{X} = -\frac{dU}{dX} + \xi \frac{\partial f}{\partial X} = -\frac{dU}{dX} - \frac{1}{m\omega^2} \frac{\partial f}{\partial X}$$

We may rewrite it as

$$m\ddot{X} = -\frac{dU_{\text{eff}}}{dX}; \quad U_{\text{eff}} = U + \frac{f^2}{2m\omega^2}.$$
Equation for Fast Motion

The original equation:

\[
\frac{\partial \phi(r)}{\partial t} = \alpha[\phi(r) - \phi(r)^3] + \beta \nabla^2 \phi(r) - \gamma \int dr' \phi(r') G(r, r') + h(t)
\]

Assumption: \( \phi(r, t) = \Phi(r, t) + \phi_0(t) \)

- \( \Phi(r, t) \): slowly varying term (space-dependent)
- \( \phi_0(t) \): rapidly oscillating term (space-independent)
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The rapidly oscillating part:

\[
\dot{\phi}_0 = \alpha(\phi_0 - \phi_0^3) - \gamma \phi_0 \int \mathrm{d}r' G(r', 0) + h_0 \sin \omega t
\]

\[ \rightarrow \phi_0 = \rho_0 \sin(\omega t + \delta) \]  
\( \rho_0 \) and \( \delta \) can be enumerated.
Approximation Methods

We propose two approximation methods to obtain the equation for slow motion [4].

1. The rapidly oscillating part is averaged out (on the basis of Kapitza’s idea).
   ⇒ Time-averaged model

2. The delay of the response to the oscillating field is considered (instead of taking a time average).
   ⇒ Phase-shifted model

Equation for Slow Motion

Dynamical equation for the slowly varying part:

1. Time-averaged model

\[
\frac{\partial \Phi(r)}{\partial t} = \alpha \left( \Phi(r) - \Phi(r)^3 \right) + \beta \nabla^2 \Phi(r) - \gamma \int dr' G(r, r')
\]

\[+ \frac{3}{2} \alpha \rho_0^2 \Phi(r)\]

2. Phase-shifted model

\[
\frac{\partial \Phi(r)}{\partial t} = \alpha \left( \Phi(r) - \Phi(r)^3 \right) + \beta \nabla^2 \Phi(r) - \gamma \int dr' G(r, r')
\]

\[-\alpha \Phi(r) \left( \Phi(r)^2 + 3\Phi(r)\rho_0 \sin \delta + 3\rho_0^2 \sin^2 \delta \right) + C'
\]

\[C' = \eta_0 \rho_0 \sin \delta - \alpha \rho_0^3 \sin^3 \delta - \omega \rho_0 \cos \delta\]
How to Discuss a Traveling Pattern

1. We consider a parallel-stripe-type solution including second harmonics:

\[ \Phi(r, t) = A_0(t) + A_1(t) \sin(kx + b(t)) + A_{21}(t) \cos[2(kx + b(t))] + A_{22}(t) \sin[2(kx + b(t))] \]
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2. Substituting the above \( \Phi(r, t) \) into the equation for slow motion, we obtain the equation for \( A_0, A_1, A_{21}, A_{22} \) and \( b \).
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3. Finding stationary points (SPs), we examine the linear stabilities at the SPs.
   If \( \dot{b} \neq 0 \) at a stable SP, the pattern travels.
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3. Finding stationary points (SPs), we examine the linear stabilities at the SPs. If \( \dot{b} \neq 0 \) at a stable SP, the pattern travels.

4. The pattern can also travel if \( \dot{b} = 0 \) at an unstable SP.
Is a Traveling Pattern Possible?

1. **Time-averaged model — impossible**

\[ \dot{b} = -3\alpha A_0 A_{22} \]

There are only SPs with \( A_0 = A_{21} = A_{22} = 0 \), and they are always stable along \( A_0 \)-axis. But we can estimate the max \( h_0 \) to observe a non-uniform pattern.

2. **Phase-shifted model — possible**

\[ \dot{b} = -3\alpha (A_0 + \rho_0 \sin \delta) A_{22} \]

There are SPs where \( A_0 + \rho_0 \sin \delta \neq 0 \) but \( A_{22} = 0 \), and they can be unstable along \( A_{22} \) in some region of \( h_0 \).
Concentric Circles

Concentric circles can appear in some cases.

- The field is very strong and the frequency is very high.
- (Assume) a strong defect at the center — The spin at the center is always up.
Above the upper red line: homogeneous pattern except for the vicinity of center.

Below the lower red line: maze or lattice patterns

Between the upper and lower red lines — Concentric circles appear.

The theoretical line above which no pattern but a homogeneous pattern appears is obtained from the time-averaged model.
Spiral Pattern under a particular field

Numerical simulations show interesting patterns under a time-periodic and spatially inhomogeneous field.

Here, we redefine the magnetic field as $h(r) = h_0 \sin \omega t$, and

$$h(r) = \begin{cases} 
    b(x^2 + y^2)/R^2 + (1 - b) & \text{when } x^2 + y^2 < R^2 \\
    0 & \text{when } x^2 + y^2 > R^2
\end{cases}$$
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\end{cases}$$

\[ R = L/4 \quad b = 0.5 \]
\[ R = 3L/8 \quad b = 0.8 \]
\[ R = 3L/4 \quad b = 0.5 \]

$L = 128$
Summary

- Under oscillating fields, a labyrinth structure changes into a parallel-stripe or lattice structure depending on the field strength and frequency.

- In some cases, we can see traveling patterns, which move very slowly compared with the time scale of the field frequency.

- Two methods were proposed to study the effects of the oscillating field.

  - Phase-shifted model explains the existence of the traveling pattern.
  
  - Time-averaged model explains the existence of the threshold of the homogeneous pattern.