Integrability of complex planar systems with homogeneous nonlinearities

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Integrability of differential systems

The integrability problem for systems of differential equations is one of the main problems in the qualitative theory of differential systems. In fact, integrability, although a rare phenomenon, is of great importance due to applications in the bifurcation theory. In the study of mathematical models it is important to detect rare systems that are integrable, since perturbations of such systems exhibit a rich behavior of bifurcations.

From the beginning of the last century many papers have been devoted to studies on the existence of a local analytic first integral in a neighborhood of a singular point for real autonomous polynomial differential systems in the plane. The most studied case is the singular point with pure imaginary eigenvalues of the matrix of linear approximation. Limiting our consideration to polynomial systems we can write such systems in the form

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \qquad \dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j. \tag{1}$$

Local integrability

Theorem (Poincaré-Lyapunov)

A system

$$\frac{du}{dt} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \quad \frac{dv}{dt} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j \tag{2}$$

has a center at the origin if and only if it admits formal first integral of the form

$$\Phi = u^2 + v^2 + \sum_{k+l \geq 3} \phi_{kl} u^k v^l.$$

The center problem

$$
\begin{aligned}
\dot{x} &= ax + by + \sum_{p+q=2}^{\infty} \alpha_{pq} x^p y^q, \\
\dot{y} &= cx + dy + \sum_{p+q=2}^{\infty} \beta_{pq} x^p y^q.
\end{aligned}
$$

$$
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \tau = a + d, \ \Delta = ad - bc
$$

$\tau = 0, \Delta > 0$
-center or focus (depending on high order terms).

Poincaré center problem

(1854-1912)

How to determine if the system

$$\dot{u} = \omega v + \sum_{p+q=2}^{n} \alpha_{pq} u^p v^q,$$
$$\dot{v} = -\omega u + \sum_{p+q=2}^{n} \beta_{pq} u^p v^q \qquad (3)$$

has center or focus at the origin?

Solved only for:

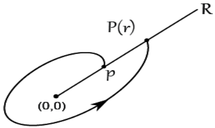
- Dulac (1908) - quadratic system
- Kapteyn (1911,1912), Frommer (1934) - real quadratic system
- Saharnikov (1948), Sibirsky (1954,1955), Malkin (1966) - correct coefficient conditions for a center for real quadratic system
- Lloyd and Pearson (1992), Sadovskii (1997) - cubic Liénard system
- Al'muhamedov (1937), Saharnikov (1950), Malkin (1964), Sadovski (1974)- linear center perturbed by cubic homogeneous polynomials
- Chavarriga and Giné (1996) - some conditions for linear center perturbed with homogeneous polynomials of degree five
- $\cdots$ some specific families.

Poincaré first return map

$$\dot{u} = \alpha u - \beta v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \quad \dot{v} = \beta u + \alpha v + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j.$$

Poincaré first return map

$$\mathcal{P}(\rho) = e^{2\pi\frac{\alpha}{\beta}} \rho + \eta_2(\alpha, \beta, \alpha_{ij}, \beta_{ij})\rho^2 + \eta_3(\alpha, \beta, \alpha_{ij}, \beta_{ij})\rho^3 + \dots.$$



$\alpha = 0, \beta = 1 \to \eta_k(\alpha_{ij}, \beta_{ij})$ are polynomials.

$$\mathcal{D}(\rho) = \mathcal{P}(\rho) - \rho = \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots$$

Coefficient η_j, $j \in N$ is called *jth Lyapunov number.*

Center

System (3) has a center at the origin if and only if all Lyapunov numbers are 0.

Complexification of a real system

We consider real system

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j = -v + U(u,v), \qquad (4)$$

$$\dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j = u + V(u,v).$$

We introduce $x = u + iv$. Then we obtain from system (4) the equation

$$\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} \overline{x}^q). \qquad (5)$$

Complexification of a real system

Then we add to (5) its complex conjugate and denote $\overline{x} = y$ in $\overline{a_{pq}} = b_{qp}$ and we obtain system

$$\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -i(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}). \tag{6}$$

The change of time $d\tau = idt$ transforms (6) to the system

$$\begin{aligned} \dot{x} &= (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P(x, y), \\ \dot{y} &= -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q(x, y), \end{aligned} \tag{7}$$

where x, y, $a_{pq}, b_{qp} \in \mathbb{C}$.

Definition of a complex center

System

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}), \tag{8}$$

has a center at the origin if it admits formal first integral of the form

$$\Psi(x, y; a_{10}, b_{10}, \ldots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j} \tag{9}$$

For complex systems

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \; \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q,$$

we can always find function
$\Psi(x, y; a_{pq}, b_{qp}) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$ such that

$$\frac{\partial \Psi}{\partial x} P + \frac{\partial \Psi}{\partial y} Q = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots , \tag{10}$$

and $g_{11}, g_{22}, \ldots$ are polynomials in the coefficients a_{pq}, b_{qp}.

Focus quantities

Polynomial g_{kk} is called *kth focus quantity.*

Bautin ideal

Bautin ideal

An ideal $\mathcal{B} = \langle g_{11}, g_{22}, \dots \rangle$ generated by focus quantities is called *Bautin ideal*. We denote by $\mathcal{B}_k$ ideal generated by first k focus quantities, $\mathcal{B}_k = \langle g_{11}, g_{22}, \dots g_{kk} \rangle$. The variety $\mathbf{V}(\mathcal{B})$ of Bautin ideal is called the *center variety* and we denote it by $V_{\mathcal{C}}$.

Center problem

Find the variety $V_{\mathcal{C}}$.

Mechanism for proving the integrability of polynomial systems

$$
\begin{aligned}
\dot{x} &= \mathcal{X}(x,y) = x - \sum_{j+k=2}^{n} a_{jk}\, x^{j+1} y^{k}, \\
\dot{y} &= \mathcal{Y}(x,y) = -y + \sum_{j+k=2}^{n} b_{jk}\, x^{j} y^{k+1}.
\end{aligned} \tag{11}
$$

- Hamiltonian systems (center at the origin)
- Time-reversibility (center at the origin)
- Darboux method
- Monodromy arguments
- Series expansions
- Blow down to a node
- Generalized time-reversibility

Integrability of systems with quadratic and cubic nonlinearities

- V. V. Amel'kin, N. A. Lukashevich, and A. P. Sadovskii (1982)
- C. Christopher, P. Mardešić, and C. Rousseau (2003)
- Y. Liu, J. Li, W. Huang (2009)
- V.G. Romanovski, D.S. Shafer (2009)
- S. Lines, D. Cozma (2013)

Integrability of systems with quartic and quintic nonlinearities

- C. X. Du, H. L. Mi, Y. R. Liu (2010)
- J. Giné, V.G. Romanovski (2010)
- B. Ferčec, X. Chen, V.G. Romanovski (2011)
- J. Giné, Z. Kadyrsizova, Y. Liu, V.G. Romanovski (2011)
- B. Ferčec, J. Giné, Y. Liu, V.G. Romanovski (2013)

Local integrability of a quintic system

The general system with quintic homogeneous nonlinearities is written as

$$\begin{aligned}
\dot{x} &= x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4 - a_{-15}y^5, \\
\dot{y} &= -y + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5,
\end{aligned} \tag{12}$$

where $x, y, a_{ij}, b_{ji} \in \mathbb{C}$.

Local integrability of a quintic system

- B. Ferčec, J. Giné, V. G. Romanovski and V. F. Edneral. Integrability of complex planar systems with homogeneous nonlinearities. Preprint 2014.

$$\dot{x} = x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4,$$
$$\dot{y} = -y + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5. \tag{13}$$

$$(C_1) \quad a_{31} = b_{13} = 1, \qquad (C_2) \quad a_{31} = 1,\ b_{13} = 0,$$
$$(C_3) \quad a_{31} = 0,\ b_{13} = 1, \qquad (C_4) \quad a_{31} = b_{13} = 0.$$

In the case when $a_{31}b_{13} \neq 0$ system (13) can be transformed into system (13) with condition (C_1) and in the case when $a_{31}b_{13} = 0$ – into a system with one of conditions $(C_2) - (C_4)$ satisfied. Hence, obtaining necessary and sufficient conditions for integrability of system (13) with one of conditions $(C_1) - (C_4)$ fulfilled we obtain the complete solution of the complex center problem for system (13).

$$
\begin{aligned}
\dot{x} &= \mathcal{X}(x,y) = x - \sum_{j+k=2}^{n} a_{jk}\, x^{j+1} y^{k}, \\
\dot{y} &= \mathcal{Y}(x,y) = -y + \sum_{j+k=2}^{n} b_{jk}\, x^{j} y^{k+1}.
\end{aligned} \tag{14}
$$

System
- with $f(x,y)=0$
  - Darboux and Liouville integrability.
  - $f(0,0)=0$ Monodromy arguments .
  - $f(0,0)\neq 0$ Proposition 1.
  - Series expansions .
- without $f(x,y)=0$
  - Reversibility .
  - $f(0,0)\neq 0$ Proposition 2.
  - Blow down to a node.

Darboux integrability

We now briefly present the method of Darboux integration for proving the existence of first integrals and integrating factors for polynomial systems of differential equations on $\mathbb{C}^2$.

Darboux integrability

We now briefly present the method of Darboux integration for proving the existence of first integrals and integrating factors for polynomial systems of differential equations on $\mathbb{C}^2$.

Consider the system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{15}$$

where $x, y \in \mathbb{C}$ and P and Q are polynomials.

Definition

The polynomial $f(x,y) \in \mathbb{C}[x,y]$ defines an *algebraic invariant curve* $f(x,y)=0$ of system (15) if there exists a polynomial $k(x,y) \in \mathbb{C}[x,y]$ such that

$$D(f) := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = kf. \tag{16}$$

The polynomial $k(x,y)$ is called *cofactor* of f.

Suppose that the curves defined by

$$f_1 = 0, \dots, f_s = 0$$

are invariant algebraic curves of system (15) with the cofactors $k_1, \dots, k_s$. If

$$\sum_{j=1}^{s} \alpha_j k_j = 0\,, \tag{17}$$

then $H = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ is a (Darboux) first integral of the system (15) and if

$$\sum_{j=1}^{s} \beta_j k_j = -P'_x - Q'_y \tag{18}$$

then $\mu = f_1^{\beta_1} \cdots f_s^{\beta_s}$ is an integrating factor of (15).

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This integrating factor is called *Darboux integrating factor*.

Time-reversibility (Reversibility)

Definition

It is said that the system

$$\dot{x} = h(x, y), \qquad \dot{y} = g(x, y) \tag{19}$$

is (time-)reversible if there is an invertible transformation R, $(x_1, y_1) = R(x, y)$, such that the system is invariant under the transformation and the time inversion $t \to -T$.

The simplest case of reversibility is when R is a linear transformation of the form

$$R : x_1 \to \gamma y, \qquad y_1 \to \gamma^{-1} x, \tag{20}$$

for some $\gamma \in \mathbb{C} \setminus \{0\}$. If a system (14) is reversible with respect to (20) then it admits a local analytic first integral of the form $\Psi = xy + h.o.t.$

Generalized reversibility

We discuss two kinds of generalized reversibility of systems (14) with homogeneous nonlinearities of degree *d* with respect to maps of the form

$$x_1 = \frac{k_1 y}{f(x,y)^{1/(d-1)}}, \qquad y_1 = \frac{k_2 x}{f(x,y)^{1/(d-1)}}. \tag{21}$$

Proposition 1 (B.F., J. Giné, V.G. Romanovski, V.F.Edneral)

Assume that for a differential system (14) there is an invertible change of coordinates $u = u(x, y), \; v = v(x, y)$, with the inverse $x = x(u, v), \; y = y(u, v)$, which brings the system to the form

$$\frac{du}{dt} = -\frac{\mathcal{X}(u, v)}{\tilde{f}(u, v)}, \qquad \frac{dv}{dt} = -\frac{\mathcal{Y}(u, v)}{\tilde{f}(u, v)}, \tag{22}$$

where $\mathcal{X}(x, y)$ and $\mathcal{Y}(x, y)$ are the same polynomials as in (14) in the variables (u, v), and

$$\left(\frac{\partial u}{\partial x}\mathcal{X}(x, y) + \frac{\partial u}{\partial y}\mathcal{Y}(x, y)\right)\Big|_{x=x(u,v),y=y(u,v)} = -\frac{\mathcal{X}(u, v)}{\tilde{f}(u, v)}, \tag{23}$$

$$\left(\frac{\partial v}{\partial x}\mathcal{X}(x, y) + \frac{\partial v}{\partial y}\mathcal{Y}(x, y)\right)\Big|_{x=x(u,v),y=y(u,v)} = -\frac{\mathcal{Y}(u, v)}{\tilde{f}(u, v)}, \tag{24}$$

$$xy = uv + h.o.t. \tag{25}$$

Then system (14) has a complex center at the origin.

Proof.

Suppose that system (14) is not integrable. Since $\mathcal{X} = x + h.o.t.$, $\mathcal{Y} = -y + h.o.t.$ there exists a formal power series $F(x,y) = xy + h.o.t$, such that

$$\left.\frac{dF(x,y)}{dt}\right|_{(14)} = \frac{\partial F(x,y)}{\partial x}\mathcal{X}(x,y) + \frac{\partial F(x,y)}{\partial y}\mathcal{Y}(x,y) = \lambda_m (xy)^m + h.o.t., \quad (26)$$

where m is a positive integer and $\lambda_m \neq 0$ is a constant. Then

$$\begin{aligned}\left.\frac{dF(x,y)}{dt}\right|_{(14)} &= \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial x}\mathcal{X}(x,y) + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial x}\mathcal{X}(x,y) \\ &+ \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial y}\mathcal{Y}(x,y) + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial y}\mathcal{Y}(x,y) \qquad (27)\\ &= \frac{-1}{\tilde{f}(u,v)}\left[\frac{\partial F(u,v)}{\partial u}\mathcal{X}(u,v) + \frac{\partial F(u,v)}{\partial v}\mathcal{Y}(u,v)\right] = -\lambda_m (xy)^m + h.o.t.,\end{aligned}$$

where the second equality holds due to (23) and (24) and the last one due to (25). From (26) and (27) we have that $\lambda_m = 0$.

$\Rightarrow$ the corresponding system (14) is integrable $\Rightarrow$a complex center at the origin.

The equation $\tilde{f}(u, v) = 0$ not necessary defines an invariant curve of system (14). However in all the examples where we apply Proposition 1 there is a unique invariant curve which is helpful in proving that system (14) has a complex center at the origin.

This approach was used first in

- B. Ferčec, J. Giné, Y. Liu, V.G. Romanovski, *Integrability conditions for Lotka-Volterra planar complex quartic systems having homogeneous nonlinearities*. Acta Appl. Math. **124** (2013), 107–122.

to prove the integrability of a certain quartic system.

We give another example of its application solving an open problem proposed in

- B. Ferčec, X. Chen and V. G. Romanovski. *Integrability conditions for complex systems with homogeneous quintic nonlinearities*. Journal of Applied Analysis and Computation (2011), no. 1, 9–20.

$$\begin{aligned}
\dot{x} &= x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4 - y^5,\\
\dot{y} &= -y + x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5.
\end{aligned} \tag{28}$$

Four subfamilies:

- $a_{40} = b_{04} = 0$
- $a_{31} = b_{13} = 0 \rightarrow$ OPEN CASE:
  $b_{22} = a_{40} + b_{04} = b_{40} = a_{04} = b_{31} + 1 = a_{13} + 1 = 0$
- $a_{13} = b_{31} = 0$
- $a_{04} = b_{40} = 0.$

$$
\begin{aligned}
\dot{x} =& x + b_{04}x^5 + x^2y^3 - y^5 = P(x,y),\\
\dot{y} =& -y + x^5 - x^3y^2 + b_{04}y^5 = Q(x,y).
\end{aligned}
\tag{29}
$$

For system (29) we have only one invariant curve of degree four $f = 1 + b_{04}(x^4 - y^4)$ which is not enough to construct a Darboux first integral or a Darboux integrating factor.

We can use Proposition 1 in order to prove that this system has first integral of the form $\psi = xy + h.o.t$. The transformation

$$
u = \frac{y}{f(x,y)^{1/4}} \quad \text{and} \quad v = \frac{x}{f(x,y)^{1/4}}
$$

satisfies (23)–(25).

Hence by Proposition 1 we see that system (29) is integrable and, therefore, has a complex center at the origin.

Sometimes the reversibility is hidden and just through a change of variables and a scaling of time can be detected. The next proposition treats this situation when the system becomes reversible with respect to involution (20) after a change of coordinate and a time rescaling.

Proposition 2 (B.F., J. Giné, V.G. Romanovski, V.F.Edneral)

Assume that by an invertible analytic transformation of the form

$$z = k_1 x + h.o.t., \qquad w = k_2 y + h.o.t. \tag{30}$$

and the time rescaling $dt = \tilde{f}(w, z)dT$ system (14) can be written in the form

$$\frac{dz}{dT} = -z(1 + h(z, w)), \qquad \frac{dw}{dT} = w(1 - h(z, w)), \tag{31}$$

where $h(z, w)$ is an analytic function of (z, w). Then system (14) has a complex center at the origin if system (31) is invariant under the transformation $z \to w$, $w \to z$ and $T \to -T$.

Proof.

System (31) is invariant under the transformation $z \to w$, $w \to z$ and $T \to -T$, that means, it is a time-reversible system. Hence, by theorem of [Romanovski&Shafer,2009] system (31) has a first integral of the form $\psi = zw + \cdots$, that is, it has a complex center at the origin and consequently, since (30) is invertible near the origin, system (14) also has a complex center at the origin.

Theorem (Romanovski&Shafer,2009)

Every time-reversible system of the form (14) has a center at the origin.

Lemma

Let f be a polynomial of the form $f(x,y) = 1 + F(x,y)$, where F is a homogeneous polynomial of degree m. Then there exists a polynomial $\tilde{f}(w,z)$ of degree m such that

$$\tilde{f}(w,z)f(x,y) \equiv 1$$

for

$$z = k_1 y/f(x,y)^{1/m} \text{ and } w = k_2 x/f(x,y)^{1/m}. \tag{32}$$

Moreover,

$$\tilde{f}(x,y) = 1 + G(x,y), \tag{33}$$

where

$$G(x,y) = -F\left(\frac{x}{k_2}, \frac{y}{k_1}\right). \tag{34}$$

Consider now a polynomial Lotka-Volterra system of the form

$$\dot{x} = x(1 + A(x, y)), \qquad \dot{y} = -y(1 + B(x, y)), \tag{35}$$

where A and B are homogeneous polynomials of degree d.

Proposition 3 (B.F., J. Giné, V.G. Romanovski, V.F.Edneral)

There exists a polynomial f of the form $f = 1 + F(x, y)$, where F is a homogeneous polynomial of degree $d - 1$ such that the change of coordinates

$$z = k_1 y / f(x, y)^{1/(d-1)} \text{ and } w = k_2 x / f(x, y)^{1/(d-1)}, \tag{36}$$

whose inverse change is given by

$$x = w/(k_2 \tilde{f}(w, z)^{1/(d-1)}) \text{ and } y = z/(k_1 \tilde{f}(w, z)^{1/(d-1)}), \tag{37}$$

where $\tilde{f}(w, z) = 1/f(x(w, z), y(w, z))$ transforms (35) to a system of the form (31).

Moreover,

$$f = 1 + \frac{A+B}{2} \tag{38}$$

and

$$h(w,z) = \frac{1}{2}\left((\hat{B} - \hat{A}) + \frac{1}{d-1}\left(\left(x\frac{\partial(A+B)}{\partial x} \right)\Big|_{x=w/k_2, y=z/k_1} (\tilde{f} + \hat{A}) - \left(y\frac{\partial(A+B)}{\partial y} \right)\Big|_{x=w/k_2, y=z/k_1} (\tilde{f} + \hat{B}) \right) \right), \tag{39}$$

where

$$\hat{A}(w,z) = A(w/k_2, z/k_1), \quad \hat{B}(w,z) = B(w/k_2, z/k_1). \tag{40}$$

As a direct corollary of Propositions 2 and 3 we obtain the following criterion for existence of a center.

Proposition 4 (B.F., J. Giné, V.G. Romanovski, V.F.Edneral)

System (35) has a complex center of the origin if

$$h(w,z) + h(z,w) \equiv 0, \tag{41}$$

where h is the function defined by (39).

$$
\begin{aligned}
\dot{x} &= x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4 - a_{-15}y^5, \\
\dot{y} &= -y + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5,
\end{aligned}
\tag{42}
$$

Theorem

System (42) with $a_{-15} = b_{5,-1} = 0$ and $a_{22} = b_{22}$ has a center at the origin if one of the following conditions holds:

1) $b_{40} = a_{04} = 3\,b_{31} - a_{31} = 3\,a_{13} - b_{13} = b_{22} = b_{04}\,a_{31}^2 + a_{40}\,b_{13}^2 = 0$,

2) $a_{40}\,a_{04} - b_{04}\,b_{40} = a_{13}\,a_{31} - b_{31}\,b_{13} = a_{31}^2\,a_{04} - b_{13}^2\,b_{40} = b_{31}\,a_{31}\,a_{04} - a_{13}\,b_{13}\,b_{40} = b_{31}^2\,a_{04} - a_{13}^2\,b_{40} = b_{04}\,a_{31}^2 - a_{40}\,b_{13}^2 = b_{31}\,b_{04}\,a_{31} - a_{13}\,a_{40}\,b_{13} = a_{13}^2\,a_{40} - b_{31}^2\,b_{04} = 0$,

$$
\begin{aligned}
&3)\ a_{40}\, a_{04} + b_{04}\, b_{40} = 3\, b_{31}\, b_{13} - a_{31}\, b_{13} + 3\, b_{04}\, b_{40} + a_{04}\, b_{40} = \\
&2\, b_{22}\, b_{13} - 3\, b_{31}\, a_{04} - a_{31}\, a_{04} = 3\, a_{13}\, a_{31} - a_{31}\, b_{13} - 3\, b_{04}\, b_{40} + a_{04}\, b_{40} = \\
&2\, b_{22}\, a_{31} - 3\, a_{13}\, b_{40} - b_{13}\, b_{40} = 9\, a_{40}\, b_{04} - 4\, a_{31}\, b_{13} + a_{04}\, b_{40} = \\
&3\, b_{31}\, b_{04} - b_{04}\, a_{31} - b_{31}\, a_{04} - a_{31}\, a_{04} = 3\, a_{13}\, b_{04} - b_{04}\, b_{13} + a_{13}\, a_{04} + b_{13}\, a_{04} = \\
&3\, b_{22}\, b_{04} - 3\, a_{13}\, b_{13} + b_{13}^2 + b_{22}\, a_{04} = 3\, b_{31}\, a_{40} - a_{40}\, a_{31} + b_{31}\, b_{40} + a_{31}\, b_{40} = \\
&3\, a_{13}\, a_{40} - a_{40}\, b_{13} - a_{13}\, b_{40} - b_{13}\, b_{40} = 3\, b_{22}\, a_{40} - 3\, b_{31}\, a_{31} + a_{31}^2 + b_{22}\, b_{40} = \\
&9\, b_{31}^2 - 6\, b_{31}\, a_{31} + a_{31}^2 + 4\, b_{22}\, b_{40} = 9\, a_{13}\, b_{31} - a_{31}\, b_{13} - 2\, a_{04}\, b_{40} = \\
&2\, b_{22}\, b_{31} + a_{13}\, b_{40} - b_{13}\, b_{40} = 9\, a_{13}^2 - 6\, a_{13}\, b_{13} + b_{13}^2 + 4\, b_{22}\, a_{04} = \\
&2\, b_{22}\, a_{13} + b_{31}\, a_{04} - a_{31}\, a_{04} = b_{22}^2 - a_{04}\, b_{40} = 4\, a_{31}\, b_{13}\, a_{04} + 9\, b_{04}^2\, b_{40} - a_{04}^2\, b_{40} = \\
&a_{31}^2\, a_{04} - 3\, a_{13}\, b_{13}\, b_{40} + b_{22}\, a_{04}\, b_{40} = 3\, b_{31}\, a_{31}\, a_{04} - 3\, a_{31}^2\, a_{04} + 9\, a_{13}\, b_{13}\, b_{40} - \\
&b_{13}^2\, b_{40} - 4\, b_{22}\, a_{04}\, b_{40} = 3\, a_{13}\, b_{13}^2 - b_{13}^3 - 3\, b_{04}\, a_{31}\, a_{04} - 3\, b_{31}\, a_{04}^2 - 2\, a_{31}\, a_{04}^2 = \\
&4\, a_{31}^3 + 9\, a_{40}^2\, b_{13} + 12\, a_{40}\, b_{13}\, b_{40} + 6\, a_{13}\, b_{40}^2 + 5\, b_{13}\, b_{40}^2 = \\
&3\, b_{04}\, a_{31}^2 + 3\, a_{40}\, b_{13}^2 + 6\, a_{13}\, b_{13}\, b_{40} + 2\, b_{13}^2\, b_{40} - b_{22}\, a_{04}\, b_{40} = \\
&3\, b_{31}\, a_{31}^2 - a_{31}^3 - 3\, a_{40}\, b_{13}\, b_{40} - 3\, a_{13}\, b_{40}^2 - 2\, b_{13}\, b_{40}^2 = \\
&9\, b_{04}^2\, a_{31} + 4\, b_{13}^3 + 12\, b_{04}\, a_{31}\, a_{04} + 6\, b_{31}\, a_{04}^2 + 5\, a_{31}\, a_{04}^2 = 16\, b_{13}^4\, a_{04} - \\
&81\, b_{04}^4\, b_{40} - 108\, b_{04}^3\, a_{04}\, b_{40} - 54\, b_{04}^2\, a_{04}^2\, b_{40} - 12\, b_{04}\, a_{04}^3\, b_{40} - a_{04}^4\, b_{40} = 0.
\end{aligned}
$$

For this system we write down the function $h(w, z)$ defined by (39). Then we see that (41) holds if and only if

$$a_{22}^2+2a_{13}a_{31}+2a_{04}a_{40}-b_{22}^2-2b_{13}b_{31}-2b_{04}b_{40} = a_{22}a_{31}k_1^2+a_{13}a_{40}k_1^2-b_{22}b_{31}k_1^2 - b_{13}b_{40}k_1^2 + a_{13}a_{22}k_2^2 + a_{04}a_{31}k_2^2 - b_{13}b_{22}k_2^2 - b_{04}b_{31}k_2^2 = a_{31}^2k_1^4 + 2a_{22}a_{40}k_1^4 - b_{31}^2k_1^4 - 2b_{22}b_{40}k_1^4 + a_{13}^2k_2^4 + 2a_{04}a_{22}k_2^4 - b_{13}^2k_2^4 - 2b_{04}b_{22}k_2^4 = a_{31}k_1^2-3b_{31}k_1^2+3a_{13}k_2^2-b_{13}k_2^2 = b_{40}k_1^4+a_{04}k_2^4 = a_{31}^2k_1^4+2a_{22}a_{40}k_1^4-b_{31}^2k_1^4-2b_{22}b_{40}k_1^4+a_{13}^2k_2^4+2a_{04}a_{22}k_2^4-b_{13}^2k_2^4-2b_{04}b_{22}k_2^4 = (a_{22} - b_{22})k_1^6k_2^6 = a_{31}a_{40}k_1^6 - b_{31}b_{40}k_1^6 + a_{04}a_{13}k_2^6 - b_{04}b_{13}k_2^6 = a_{31}k_1^2 - 3b_{31}k_1^2 + 3a_{13}k_2^2 - b_{13}k_2^2 = a_{40}^2k_1^8 - b_{40}^2k_1^8 + a_{04}^2k_2^8 - b_{04}^2k_2^8 = a_{40}^2k_1^8 - b_{40}^2k_1^8 + a_{04}^2k_2^8 - b_{04}^2k_2^8b_{40}k_1^4 + a_{04}k_2^4 = 0.$$

Since k_1 and k_2 should be different from 0, we add to the polynomials defining the system given above the polynomials $1 - \alpha k_1$ and $1 - \beta k_2$ obtaining an ideal which we denote by I. Then we compute the forth elimination ideal $I_4 = I \cap \mathbb{Q}[a_{40}, \dots, a_{04}, b_{40}, \dots, b_{04}]$ (we did this with the routine `eliminate` of the computer algebra SINGULAR).

Further computations became very laborious, so to simplify them without lose of generality we set $a_{22} = b_{22}$ (since the first non-zero focus quantity is $g_{2,2} = a_{22} - b_{22}$). Then, with the routine `minAssGTZ` of the `primdec` library of SINGULAR we find that the set of common zeros of the polynomials of the ideal I_4 consists of 3 components presented in the statement of the theorem.

Similar approach can be applied to any system with homogeneous nonlinearities (not only to Lotka-Volterra systems). However in the general case we cannot transform the system to the form (31) and we were not able to obtain a generalization of Proposition 3. We demonstrate how the approach works for general systems with homogeneous nonlinearities using as example the quadratic system

$$\begin{aligned} \dot{x} &= x + a_{10}x^2 + a_{01}xy + a_{-12}y^2, \\ \dot{y} &= -y + b_{10}xy + b_{01}y^2 + b_{2,-1}x^2. \end{aligned} \tag{43}$$

Let $f = 1 + c_1 x + c_2 y$ and $\tilde{f} = 1 - c_1 w/k_2 - c_2 z/k_1$. Then $f\tilde{f}$ evaluated at

$$w = \frac{xk_2}{f}, \quad z = \frac{yk_1}{f} \tag{44}$$

is identically equal to one. Applying the inverse of (44) to system (43) we obtain a cubic system. The obtained system is time-reversible if conditions of Theorem 6 of [C.Christopher&C.Rousseau (2004)] are fulfilled. We plug in the coefficients of the obtained systems to the polynomials of Theorem 6 , then, similarly as above, we add the polynomials $1 - k_1\alpha,\ 1 - k_2\beta$, then eliminate from the obtained ideal $\alpha, \beta, k_1, k_2, c_1$ and c_2.

Computing the minimal associated primes of the obtained ideal we obtain two ideals:

$$J_1 = \langle a_{10}a_{01} - b_{10}b_{01}, a_{-12}b_{10}^3 - b_{2,-1}a_{01}^3, a_{10}^3a_{-12} - b_{2,-1}b_{01}^3,$$
$$a_{10}a_{-12}b_{10}^2 - a_{01}^2b_{2,-1}b_{01}, a_{10}^2a_{-12}b_{10} - a_{01}b_{2,-1}b_{01}^2\rangle,$$

and

$J_2 = \langle b_{10}, a_{01}, a_{10}^3a_{-12} - 6a_{10}a_{-12}b_{01}b_{2,-1} + b_{01}^3b_{2,-1} + 8a_{-12}^2b_{2,-1}^2\rangle$.
As it is well-known [Romanosvki& Shafer (2009)] the variety of J_1 is the component of the center variety of system (43) corresponding to systems reversible with respect to the transformation

$$x \mapsto \gamma y,\ y \mapsto \gamma^{-1}x,\ t \mapsto -t,$$

and the variety of J_2 is the subset of the component of the center variety corresponding to Darboux integrable systems with 3 invariant lines.

Local integrability of a quintic system

$$\dot{x} = x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4,$$
$$\dot{y} = -y + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5. \tag{45}$$

$(C_1)\quad a_{31} = b_{13} = 1, \qquad (C_2)\quad a_{31} = 1,\ b_{13} = 0,$

$(C_3)\quad a_{31} = 0,\ b_{13} = 1, \qquad (C_4)\quad a_{31} = b_{13} = 0.$

Theorem 1

For system (45) with condition (C_1) and $b_{22} = a_{22}$ the following conditions are necessary for existence of the complex center at the origin:

1) $b_{5,-1} = a_{13} - b_{31} = b_{31}a_{04} - 3a_{04} - b_{04} + 3b_{04}b_{31} = 3b_{31}a_{40} - a_{40} + b_{31}b_{40} - 3b_{40} = a_{13} + a_{04}a_{40} - b_{31} - b_{04}b_{40} = 0;$

$\vdots$

14) $5b_{22} - 16b_{04} = 5b_{31} + 3 = 9b_{04} + 5b_{40} = 25b_{5,-1} + 6 = a_{04} + 7b_{04} = a_{13} + 1 = 25a_{40} - 3b_{04} = 12b_{04}^2 - 5 = 0;$

$\vdots$

18) $3b_{22} - 4b_{04} = 3b_{40} - 5b_{04} = 2b_{5,-1} + 1 = a_{04} - 3b_{04} = b_{31} = 2a_{13} - 1 = a_{40} + 3b_{04} = 64b_{04}^2 - 3 = 0;$

19) $b_{04} = b_{22} = 3b_{31} - 1 = b_{40} = a_{04} = 3a_{13} - 1 = a_{40} = 0.$

Moreover, if one of conditions 1)-7), 9)-19) is fulfilled then the corresponding system has a complex center at the origin.

Darboux integrability

Case 14

$$
\begin{aligned}
f_1 &= x,\\
f_2 &= 1 - P_8(x,y)\\
f_3 &= Q_8(x,y),\\
f_4 &= 1 - R_8(x,y)\\
f_5 &= y + H_5(x,y).
\end{aligned}
$$

$$\Downarrow$$

DARBOUX FIRST INTEGRAL:

$$\Psi(x,y) = f_1 f_2 f_5 f_3^{-\frac{9}{2}} f_4^2 = xy + h.o.t.$$

Case 19

The corresponding system is

$$\begin{aligned}\dot{x} &= x - x^4 y - \frac{x^2 y^3}{3},\\ \dot{y} &= -y + b_{5,-1} x^5 + \frac{x^3 y^2}{3} + x y^4.\end{aligned} \tag{46}$$

We introduce the change of variables $u = x^3 y$ and $v = x^8$, whose inverse change is

$$x = v^{\frac{1}{8}}, \quad y = v^{-\frac{3}{8}} u,$$

which transforms, after a re-scaling of time, system (46) into

$$\begin{aligned}\dot{u} &= u + \frac{b_{5,-1} v}{2} - \frac{4u^2}{3},\\ \dot{v} &= 4v - 4uv - \frac{4u^3}{3}.\end{aligned} \tag{47}$$

System (47) has a resonant node at the origin. The transformation $u = X + b_{5,-1}Y,\ v = 6Y$ brings (47) to the form

$$
\begin{aligned}
\dot{X} =& X + \frac{2b_{5,-1}^4 Y^3}{9} + \frac{2}{3} b_{5,-1}^3 XY^2 + \frac{2}{3} b_{5,-1}^2 X^2 Y \\
&+ \frac{8b_{5,-1}^2 Y^2}{3} + \frac{2b_{5,-1}X^3}{9} + \frac{4b_{5,-1}XY}{3} - \frac{4X^2}{3} \\
\dot{Y} =& 4Y - \frac{2b_{5,-1}^3 Y^3}{9} - \frac{2}{3} b_{5,-1}^2 XY^2 - \frac{2}{3} b_{5,-1} X^2 Y \\
&- 4b_{5,-1}Y^2 - \frac{2X^3}{9} - 4XY.
\end{aligned}
\tag{48}
$$

Poincaré-Lyapunov normal form theory

An analytic system

$$\dot{X} = X + \sum_{j+k=2}^{\infty} U_{jk} X^j Y^k, \quad \dot{Y} = nY + \sum_{j+k=2}^{\infty} V_{jk} X^j Y^k,$$

by a convergent transformation

$$\xi = X + \sum_{j+k=2}^{\infty} \alpha_{jk} X^j Y^k, \quad \eta = Y + \sum_{j+k=2}^{\infty} \beta_{jk} X^j Y^k,$$

can be brought to the normal form

$$\dot{\xi} = -\xi, \quad \dot{\eta} = -n\eta + a\xi^n. \tag{49}$$

System (47) is linearizable if and only if the resonant monomial $a\xi^4$ in the formal normal form is zero. Calculations of the normal form show that for system (48) this is the case, that is the normal form of the system is linear.

Normalizing transformation

$$X = X_1 - \frac{4}{3} X_1^2 + \frac{1}{3} b_{5,-1} X_1 Y_1 + \frac{8}{21} b_{5,-1}^2 Y_1^2 + O(X_1^2, X_1 Y_1, Y_1^2),$$
$$Y = Y_1 - 4 X_1 Y_1 - b_{5,-1} Y_1^2 + O(X_1^2, X_1 Y_1, Y_1^2), \tag{50}$$

$$\Downarrow$$

Linear system

$\dot{X}_1 = X_1,\ \dot{Y}_1 = 4 Y_1$

FIRST INTEGRAL: X_1^4 / Y_1