Center, weak focus and cyclicity problems for planar systems

Joan Torregrosa

Universitat Autònoma de Barcelona

http://www.gsd.uab.cat

Maribor, April, 2015

References

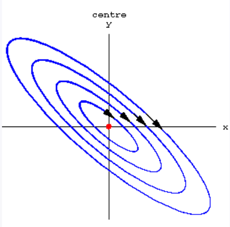
The talk is based in the next three papers:

- A. Gasull, J. Giné & J. Torregrosa. "Center problem for systems with two monomial nonlinearities". *Preprint* (2014). Submitted.
- H. Liang & J. Torregrosa. "Weak foci of high order and cyclicity". *Preprint* (2015). Submitted.
- H. Liang & J. Torregrosa. "Parallelization of the computation of Lyapunov constants and cyclicity of centers". Work in progress.

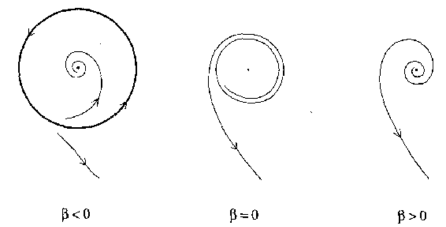
Outline

Plan

Centers and weak foci (Hopf and Degenerate-Hopf bifurcations)

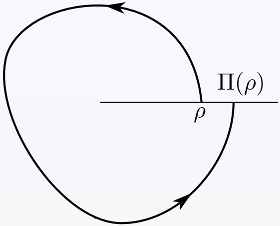


Centers



Hopf bifurcation

Main Tool: Return map



Return map

The center-focus problem and related problems

Definition

If $V_K \neq 0$ and

$$\Pi(\rho) - \rho = V_K \rho^K + O(\rho^{K+1})$$

for $\rho > 0$ close to zero, then V_K is called the K-th Lyapunov constant.

Related Problems

- *Characterization of Centers*
- *Maximum order of a Weak Focus*
- *Local Cyclicity*

The center-focus problem and related problems

Definition

If $V_K \neq 0$ and

$$\Pi(\rho) - \rho = V_K \rho^K + O(\rho^{K+1})$$

for $\rho > 0$ close to zero, then V_K is called the K-th Lyapunov constant.

Related Problems

- *Characterization of Centers*
- *Maximum order of a Weak Focus*
- *Local Cyclicity*

Lyapunov constants

For differential systems, an elementary singular point is of center-focus type if $\mathrm{tr}DX(x_0) = 0$ and $\det DX(x_0) < 0$. Then after a translation and a change of time the system writes as:

$$(x', y') = (-y + P(x, y), x + Q(x, y))$$

and, in complex coordinates ($z = x + iy$),

$$z' = i\,z + \sum_{k+\ell=m} r_{k,\ell}\, z^k \bar{z}^\ell.$$

- $V_{2K} = 0$ for all K.
- Quasihomogeneity and zero weight: $V_{2K+1}(\lambda^{-k+\ell+1} r_{k,\ell}, \lambda^{k-\ell-1} \bar{r}_{k,\ell}) = V_{2K+1}(r_{k,\ell}, \bar{r}_{k,\ell})$.
- Quasihomogeneity and quasidegree: $V_{2K+1}(\lambda^{k+\ell-1} r_{k,\ell}, \lambda^{k+\ell-1} \bar{r}_{k,\ell}) = \lambda^{2K} V_{2K+1}(r_{k,\ell}, \bar{r}_{k,\ell})$.
- $V_{2K+1} = \mathrm{Re}(V^o_{2K+1}) + \mathrm{Im}(V^e_{2K+1})$.

Lyapunov constants

For differential systems, an elementary singular point is of center-focus type if $\operatorname{tr} DX(x_0) = 0$ and $\det DX(x_0) < 0$. Then after a translation and a change of time the system writes as:

$$(x', y') = (-y + P(x, y), x + Q(x, y))$$

and, in complex coordinates ($z = x + iy$),

$$z' = i\, z + \sum_{k+\ell=m} r_{k,\ell}\, z^k \bar{z}^\ell.$$

- $V_{2K} = 0$ for all K.
- Quasihomogeneity and zero weight: $V_{2K+1}(\lambda^{-k+\ell+1} r_{k,\ell}, \lambda^{k-\ell-1} \bar{r}_{k,\ell}) = V_{2K+1}(r_{k,\ell}, \bar{r}_{k,\ell})$.
- Quasihomogeneity and quasidegree: $V_{2K+1}(\lambda^{k+\ell-1} r_{k,\ell}, \lambda^{k+\ell-1} \bar{r}_{k,\ell}) = \lambda^{2K} V_{2K+1}(r_{k,\ell}, \bar{r}_{k,\ell})$.
- $V_{2K+1} = \operatorname{Re}(V^o_{2K+1}) + \operatorname{Im}(V^e_{2K+1})$.

First Lyapunov constants

$$
\begin{aligned}
V_3 =& \mathrm{Re}(r_{2,1}) - \mathrm{Im}(r_{2,0} r_{1,1}).\\
V_5 =& \mathrm{Re}(r_{3,2}) + \frac{1}{3}\mathrm{Im}(-\bar{r}_{1,3} r_{0,2} - 3\bar{r}_{2,0} r_{3,1} - 3\bar{r}_{2,2} r_{1,1} - 4 r_{0,2} r_{4,0}\\
& - 6 r_{1,1} r_{3,1} - 3 r_{1,2} r_{3,0}) + \frac{1}{3}\mathrm{Re}(2\bar{r}_{0,2} r_{0,3} r_{2,0} + 3\bar{r}_{0,2} r_{1,1} r_{1,2}\\
& + \bar{r}_{0,3} r_{0,2} r_{1,1} + 5\bar{r}_{1,1} r_{0,2} r_{3,0} - 15\bar{r}_{1,1} r_{1,1} r_{2,1} + 3\bar{r}_{1,1} r_{1,2} r_{2,0}\\
& + 2\bar{r}_{1,2} r_{0,2} r_{2,0} - 3\bar{r}_{2,0} r_{1,1} r_{3,0} - 30\bar{r}_{2,0} r_{2,0} r_{2,1} - 21\bar{r}_{2,1} r_{1,1} r_{2,0}\\
& - 2 r_{0,2} r_{2,0} r_{3,0} - 6 r_{1,1}^2 r_{3,0} - 24 r_{1,1} r_{2,0} r_{2,1})\\
& + \frac{1}{3}\mathrm{Im}(4\bar{r}_{0,2}\bar{r}_{1,1}\bar{r}_{2,0} r_{0,2} - 2\bar{r}_{0,2} r_{1,1}^3 + 3\bar{r}_{1,1}^2 r_{0,2} r_{2,0} - 2\bar{r}_{1,1} r_{0,2} r_{2,0}^2\\
& + 15\bar{r}_{1,1} r_{1,1}^2 r_{2,0} + 30\bar{r}_{2,0} r_{1,1} r_{2,0}^2 + 24 r_{1,1}^2 r_{2,0}^2).
\end{aligned}
$$

Number of monomials $N_3 = 4$, $N_5 = 54$, $N_7 = 526(0.2\text{s})$, $N_9 = 3800(9\text{s})$, $N_{11} = 23442(14\text{m})$,

First Lyapunov constants

$$
\begin{aligned}
V_3 =& \mathrm{Re}(r_{2,1}) - \mathrm{Im}(r_{2,0}r_{1,1}).\\
V_5 =& \mathrm{Re}(r_{3,2}) + \frac{1}{3}\mathrm{Im}(-\bar r_{1,3}r_{0,2} - 3\bar r_{2,0}r_{3,1} - 3\bar r_{2,2}r_{1,1} - 4r_{0,2}r_{4,0}\\
& - 6r_{1,1}r_{3,1} - 3r_{1,2}r_{3,0}) + \frac{1}{3}\mathrm{Re}(2\bar r_{0,2}r_{0,3}r_{2,0} + 3\bar r_{0,2}r_{1,1}r_{1,2}\\
& + \bar r_{0,3}r_{0,2}r_{1,1} + 5\bar r_{1,1}r_{0,2}r_{3,0} - 15\bar r_{1,1}r_{1,1}r_{2,1} + 3\bar r_{1,1}r_{1,2}r_{2,0}\\
& + 2\bar r_{1,2}r_{0,2}r_{2,0} - 3\bar r_{2,0}r_{1,1}r_{3,0} - 30\bar r_{2,0}r_{2,0}r_{2,1} - 21\bar r_{2,1}r_{1,1}r_{2,0}\\
& - 2r_{0,2}r_{2,0}r_{3,0} - 6r_{1,1}^2r_{3,0} - 24r_{1,1}r_{2,0}r_{2,1})\\
& + \frac{1}{3}\mathrm{Im}(4\bar r_{0,2}\bar r_{1,1}\bar r_{2,0}r_{0,2} - 2\bar r_{0,2}r_{1,1}^3 + 3\bar r_{1,1}^2r_{0,2}r_{2,0} - 2\bar r_{1,1}r_{0,2}r_{2,0}^2\\
& + 15\bar r_{1,1}r_{1,1}^2r_{2,0} + 30\bar r_{2,0}r_{1,1}r_{2,0}^2 + 24r_{1,1}^2r_{2,0}^2).
\end{aligned}
$$

Number of monomials $N_3 = 4$, $N_5 = 54$, $N_7 = 526(0.2s)$, $N_9 = 3800(9s)$, $N_{11} = 23442(14m)$,

Plan

The center problem

Center Problem

$$\{\Pi(\rho) \equiv \rho\} \Leftrightarrow \{V_3 = 0, V_5 = 0, \dots, V_{2K+1} = 0, \dots\}$$

General Problems / Family Problems

- Finiteness problem $\Leftrightarrow$ Hilbert's Basis Theorem
- Computational difficulties:
  - Explicit computation
  - Solution of polynomial system of equations of high degree
  - Radicality
  - $\mathbb{R}$ versus $\mathbb{C}$
- Why is it a center? (First integral, Hamiltonian, Darboux, reversible, symmetry,...)

The center problem

Center Problem

$$\{\Pi(\rho) \equiv \rho\} \Leftrightarrow \{V_3 = 0, V_5 = 0, \ldots, V_{2K+1} = 0, \ldots\}$$

General Problems / Family Problems

- Finiteness problem $\Leftrightarrow$ Hilbert's Basis Theorem
- Computational difficulties:
  - Explicit computation
  - Solution of polynomial system of equations of high degree
  - Radicality
  - $\mathbb{R}$ versus $\mathbb{C}$
- Why is it a center? (First integral, Hamiltonian, Darboux, reversible, symmetry,. . .)

The center problem

Center Problem

$$\{\Pi(\rho) \equiv \rho\} \Leftrightarrow \{V_3 = 0, V_5 = 0, \ldots, V_{2K+1} = 0, \ldots\}$$

General Problems / Family Problems

- Finiteness problem $\Leftrightarrow$ Hilbert's Basis Theorem
- Computational difficulties:
  - Explicit computation
  - Solution of polynomial system of equations of high degree
  - Radicality
  - $\mathbb{R}$ versus $\mathbb{C}$
- Why is it a center? (First integral, Hamiltonian, Darboux, reversible, symmetry,. . .)

The center problem

Center Problem

$$\{\Pi(\rho) \equiv \rho\} \Leftrightarrow \{V_3 = 0, V_5 = 0, \dots, V_{2K+1} = 0, \dots\}$$

General Problems / Family Problems

- Finiteness problem $\Leftrightarrow$ Hilbert's Basis Theorem
- Computational difficulties:
  - Explicit computation
  - Solution of polynomial system of equations of high degree
  - Radicality
  - $\mathbb{R}$ versus $\mathbb{C}$
- Why is it a center? (First integral, Hamiltonian, Darboux, reversible, symmetry,. . .)

The center problem

Center Problem

$$\{\Pi(\rho) \equiv \rho\} \Leftrightarrow \{V_3 = 0, V_5 = 0, \ldots, V_{2K+1} = 0, \ldots\}$$

General Problems / Family Problems

- Finiteness problem $\Leftrightarrow$ Hilbert's Basis Theorem
- Computational difficulties:
  - Explicit computation
  - Solution of polynomial system of equations of high degree
  - Radicality
  - $\mathbb{R}$ versus $\mathbb{C}$
- Why is it a center? (First integral, Hamiltonian, Darboux, reversible, symmetry,. . .)

Systems with few monomials

Consider the family of differential equations

$$\dot{z} = iz + Az^k \bar{z}^\ell + Bz^m \bar{z}^n \tag{1}$$

where $k + \ell \leq m + n$, $(k, \ell) \neq (m, n)$ and $A, B \in \mathbb{C}$.
The integer values

$$\alpha = k - \ell - 1, \quad \beta = m - n - 1,$$

play a key role in the study. In particular when $\alpha = 0$ (resp. $\beta = 0$) the monomial $z^k \bar{z}^\ell$ (resp. $z^m \bar{z}^n$) appears as a resonant monomial in the Poincaré normal form.

Centers for systems with few monomials

$$\alpha = k - \ell - 1, \quad \beta = m - n - 1,$$

Theorem (GasGinTor2014)

The origin of equation $\dot{z} = iz + Az^k\bar{z}^\ell + Bz^m\bar{z}^n$ *is a center when one of the following (nonexclusive) conditions hold:*

(a) $k = n = 2$ *and* $\ell = m = 0$ *(quadratic Darboux centers).*

(b) $\ell = n = 0$ *(holomorphic centers).*

(c) $A = -\bar{A}\,\mathrm{e}^{i\,\alpha\,\varphi}$ *and* $B = -\bar{B}\,\mathrm{e}^{i\,\beta\,\varphi}$ *for some* $\varphi \in \mathbb{R}$ *(reversible centers).*

(d) $k = m$ *and* $(\ell - n)\alpha \neq 0$ *(Hamiltonian or new Darboux centers).*

A. Gasull, J. Giné & J. Torregrosa. "Center problem for systems with two monomial nonlinearities". *Preprint* (2014).

New family of centers

Theorem (GasGinTor2014)

Consider the differential equation $z' = i\,z + z^k f(\bar{z})$ *where* $f(\bar{z}) = \sum_{\ell \geq 0} f_\ell \bar{z}^\ell$ *for* $k \geq 0$*, and* $z^k f(\bar{z})$ *starts at the origin at least with second degree terms. Then the origin is a center if and only if either* $k \in \{0, 1\}$ *or* $k > 1$ *and* $\mathrm{Re}(f_{k-1}) = 0$*.*

Proof.

1. For $k = 0$ the system is Hamiltonian.
2. For $k > 0$, $U(z, \bar{z}) = z\bar{z} = 0$ is an invariant algebraic curve: $\dot{U}(z, \bar{z}) = \dot{z}\bar{z} + z\dot{\bar{z}} = 2\mathrm{Re}(z^{k-1}F(\bar{z}))\,U(z, \bar{z})$.
3. The function $U^{-k}(z, \bar{z}) = (z\bar{z})^{-k}$ is an integrating factor of the differential equation. This is precisely the definition of a Darboux integrable equation.

□

New family of centers

Theorem (GasGinTor2014)

Consider the differential equation $z' = i\,z + z^k f(\bar{z})$ where $f(\bar{z}) = \sum_{\ell \geq 0} f_\ell \bar{z}^\ell$ for $k \geq 0$, and $z^k f(\bar{z})$ starts at the origin at least with second degree terms. Then the origin is a center if and only if either $k \in \{0, 1\}$ or $k > 1$ and $\mathrm{Re}(f_{k-1}) = 0$.

Proof.

1. For $k = 0$ the system is Hamiltonian.
2. For $k > 0$, $U(z, \bar{z}) = z\bar{z} = 0$ is an invariant algebraic curve: $\dot{U}(z, \bar{z}) = \dot{z}\bar{z} + z\dot{\bar{z}} = 2\mathrm{Re}\big(z^{k-1} F(\bar{z})\big)\, U(z, \bar{z})$.
3. The function $U^{-k}(z, \bar{z}) = (z\bar{z})^{-k}$ is an integrating factor of the differential equation. This is precisely the definition of a Darboux integrable equation.

□

Centers for systems with few monomials

Theorem (GasGinTor2014)

For equation $\dot{z} = iz + Az^k\bar{z}^\ell + Bz^m\bar{z}^n$, the list of centers is complete:

(a) *when* $A\,B = 0$;

(b) *when* $\alpha\,\beta = 0$;

(c) *when* $(\alpha+\beta)(\alpha-\beta) = 0$;

(d) *when* k, ℓ, m *and* n *satisfy* $p\alpha + q\beta = 0$, $(k+\ell-1)Q-(m+n-1)P = 0$, *for some* P, Q, p *and* q, *where* $P \le Q$ *and* $\mathcal{N}(P,Q)$ *are given in the Table and* $(p,q) \in \mathbb{N}\times\mathbb{Z}$ *are such that* $pP + |q|Q \le \mathcal{N}(P,Q)$;

(e) *when the nonlinearities are homogeneous* $(k+\ell = m+n = d)$ *and either* d *is even and* $d \le 34$ *or* d *is odd and* $d \le 57$;

(f) *when* $4 \le k+\ell+m+n \le 36$.

$P\backslash Q$	1	2	3	4	5	6
1	8	10	13	13	15	15
2	-	-	19	-	19	-
3	-	-	-	23	23	-

Values of $\mathcal{N}(P,Q)$ for $P \le Q$ and coprime P and Q

Centers for systems with few monomials

The center-focus problem for equation (1) is totally solved when $\alpha\beta = 0$ or $AB = 0$. Consequently, we can reduce our problem to

$$\dot{z} = iz + z^k \bar{z}^\ell + C z^m \bar{z}^n, \qquad (2)$$

with $k + \ell \leq m + n$, $(k, \ell) \neq (m, n)$, $\alpha\beta \neq 0$ and $0 \neq C \in \mathbb{C}$.

The characterization of the reversible centers given in the above result reduces to

$$C^{|q|} + (-1)^{p+|q|+1} \bar{C}^{|q|} = 0,$$

where $(p, q) \in \mathbb{N} \times \mathbb{Z}$ are the coprime values and $p\alpha + q\beta = 0$.

Problems (GasGinTor2014)

- *Is the list of centers of equation with two monomials exhaustive?*
- *In the particular case of homogeneous nonlinearities, is it true that when $k + \ell = m + n \geq 3$ all the centers are reversible?*

Centers for systems with few monomials

The center-focus problem for equation (1) is totally solved when $\alpha\beta = 0$ or $AB = 0$. Consequently, we can reduce our problem to

$$\dot{z} = iz + z^k \bar{z}^\ell + C z^m \bar{z}^n, \tag{2}$$

with $k + \ell \leq m + n$, $(k, \ell) \neq (m, n)$, $\alpha\beta \neq 0$ and $0 \neq C \in \mathbb{C}$.
The characterization of the reversible centers given in the above result reduces to

$$C^{|q|} + (-1)^{p+|q|+1} \bar{C}^{|q|} = 0,$$

where $(p, q) \in \mathbb{N} \times \mathbb{Z}$ are the coprime values and $p\alpha + q\beta = 0$.

Problems (GasGinTor2014)

- *Is the list of centers of equation with two monomials exhaustive?*
- *In the particular case of homogeneous nonlinearities, is it true that when $k + \ell = m + n \geq 3$ all the centers are reversible?*

Centers for systems with few monomials

The center-focus problem for equation (1) is totally solved when $\alpha\beta = 0$ or $AB = 0$. Consequently, we can reduce our problem to

$$\dot{z} = iz + z^k\bar{z}^\ell + Cz^m\bar{z}^n, \tag{2}$$

with $k+\ell \leq m+n$, $(k,\ell) \neq (m,n)$, $\alpha\beta \neq 0$ and $0 \neq C \in \mathbb{C}$.
The characterization of the reversible centers given in the above result reduces to

$$C^{|q|} + (-1)^{p+|q|+1}\bar{C}^{|q|} = 0,$$

where $(p,q) \in \mathbb{N} \times \mathbb{Z}$ are the coprime values and $p\alpha + q\beta = 0$.

Problems (GasGinTor2014)

- *Is the list of centers of equation with two monomials exhaustive?*
- *In the particular case of homogeneous nonlinearities, is it true that when $k+\ell = m+n \geq 3$ all the centers are reversible?*

Plan

Order of a weak focus

Order of a weak focus

The order of a weak focus is the smallest value of K such that

$$\Pi(\rho) - \rho = V_{2K+1}\rho^{2K+1} + O(\rho^{2K+2}).$$

Maximum order problem

For a given family of polynomial vector fields which is the highest value for the order of a weak focus?

Order of a weak focus

Order of a weak focus

The order of a weak focus is the smallest value of K such that

$$\Pi(\rho) - \rho = V_{2K+1}\rho^{2K+1} + O(\rho^{2K+2}).$$

Maximum order problem

For a given family of polynomial vector fields which is the highest value for the order of a weak focus?

High order Weak foci of polynomial systems of degree n

Theorem (QiuYan2010)

For every even (odd) n, $n \geq 2$ ($n \geq 3$), there are systems $z' = iz + P_n(z, \bar{z})$ (P_n a homogeneous polynomial of degree n) with a weak focus at the origin with order no less than $n^2 - 1$ ($(n^2 - 1)/2$).

Theorem (LliRab2012)

For every even n, $n > 3$, there are systems $z' = iz(1 - (z + \bar{z})^n/2^n - \alpha(z - \bar{z})^n/(2i)^n)$ with a weak focus of order $n^2 - 1$ at the origin. (For odd n the order is $(n^2 - 1)/2$, changing the eq.)

- Y. Qiu & J. Yang. "On the focus order of planar polynomial differential equations". *J. Differential Equations* 246 (2009) 3361–3379.
- J. Llibre & R. Rabanal. "Planar real polynomial differerential systems of degree $n > 3$ having a weak focus of high order". *Rocky Mountain J. of Math.* 42 (2012) 657–693.

Simple equations with weak foci of high order?

Problem (GasGinTor2014)

There exists C such that the origin of

$$z' = i\,z + z^n + C\,z^{n-1}\bar{z}$$

is a weak focus of order $K = (n+2)(n-1)/2$? ($V_{2K+1} \neq 0$)

True up to (odd) $n = 89$. Order 4004. (2 days of CPU time).

$V_3 = V_5 = \ldots = V_{7831} = 0$, $V_{7833} = D_1(E_1 C\bar{C} - E_2)(C^{44} + \bar{C}^{44})\pi$,

$V_{7835} = \ldots = V_{8007} = 0$, $V_{8009} = -D_2(C^{44} + \bar{C}^{44})\pi$.

$D_1 = N_{1225}/N_{220}$, $E_1 = N_{157}$, $E_2 = M_{155}$, $D_2 = N_{2089}/N_{903}$.

Simple equations with weak foci of high order?

Problem (GasGinTor2014)

There exists C such that the origin of

$$z' = i\,z + z^n + C\,z^{n-1}\bar{z}$$

is a weak focus of order $K = (n+2)(n-1)/2$*?* $(V_{2K+1} \neq 0)$

True up to (odd) $n = 89$. Order 4004. (2 days of CPU time).

$V_3 = V_5 = \ldots = V_{7831} = 0,\ V_{7833} = D_1(E_1 C\bar{C} - E_2)(C^{44} + \bar{C}^{44})\pi,$

$V_{7835} = \ldots = V_{8007} = 0,\ V_{8009} = -D_2(C^{44} + \bar{C}^{44})\pi.$

$D_1 = N_{1225}/N_{220},\ E_1 = N_{157},\ E_2 = M_{155},\ D_2 = N_{2089}/N_{903}.$

Simple examples of order $O(n^2)$

Theorem (LiaTor2015)

For every degree n the origin of equation

$$z' = i\,z + \bar{z}^{n-1} + z^n$$

is a weak focus of order $(n-1)^2$.

Proof. The first nonvanishing Lyapunov constant is $V_{2(n-1)^2+1}$,

$$\Pi(\rho) - \rho = \frac{(-1)^{n+1}(2n-1)!\pi}{n!(n-1)!2^{2n-3}}\rho^{2(n-1)^2+1} + O(\rho^{2(n-1)^2+2}).$$

□

H. Liang & J. Torregrosa. “Weak foci of high order and cyclicity”. *Preprint* (2015). Submitted.

For even (low) degree

Proposition (QiuYan2010,LiaTor2015)

For $n(even) \in \{4, \ldots, 18, 20, \ldots, 32\}$ consider the system of degree n

$$z' = i\,z - \frac{n}{n-2}\,z^n + z\bar{z}^{n-1} + iC_n\,\bar{z}^n.$$

Then there exists a number C_n such that the above system has a weak focus at the origin of order $n^2 + n - 2$.

Y. Qiu & J. Yang. "On the focus order of planar polynomial differential equations". *J. Differential Equations* 246 (2009) 3361–3379.

H. Liang & J. Torregrosa. "Weak foci of high order and cyclicity". *Preprint* (2015). Submitted.

For even (low) degree

Proposition (QiuYan2010,LiaTor2015)

For $n(\text{even}) \in \{4, \ldots, 18, 20, \ldots, 32\}$ *consider the system of degree* n

$$z' = i\,z - \frac{n}{n-2}\,z^n + z\bar{z}^{n-1} + iC_n\,\bar{z}^n.$$

Then there exists a number C_n *such that the above system has a weak focus at the origin of order* $n^2 + n - 2$.

Y. Qiu & J. Yang. "On the focus order of planar polynomial differential equations". *J. Differential Equations* 246 (2009) 3361–3379.

H. Liang & J. Torregrosa. "Weak foci of high order and cyclicity". *Preprint* (2015). Submitted.

Weak focus of systems of degree 4 and 5

Proposition (HuaWanWanYan2008)

The next systems of degree 4 and 5 have a weak focus of order 18 at the origin.

$$z' = i\,z + 2i\,z^4 + i\,z\bar{z}^3 + \sqrt{\frac{52278}{20723}}\,\bar{z}^4,$$

$$z' = i\,z + 3\,z^5 + \sqrt{\frac{20(c+3)}{9c^2-15}}\,z^4\bar{z} + z\bar{z}^4 + \sqrt{\frac{20c^2(c+3)}{9c^2-15}}\,\bar{z}^5,$$

where c is the root between $(-3,-5/3)$ of the equation
$4155c^6 - 10716c^5 - 63285c^4 - 18070c^3 + 168075c^2 + 205450c + 60375 = 0.$

J. Huang, F. Wang, L. Wang & J. Yang. "A quartic system and a quintic system with fine focus of order 18". *Bull. Sci. Math.* 132 (2008) 205–217.

Plan

Order of weak foci and cyclicity

Cyclicity of a singular point

For a given family of polynomial vector fields which is the maximum number of limit cycles that bifurcate from an elementary weak focus or an elementary center?

Problem

Fixed the degree or the family, the number of limit cycles coincides with the order of the weak focus ?

Theorem

For a general system, the number of limit cycles that bifurcate from a weak focus of order K $(V_{2K+1} \neq 0)$ is K.

R. Roussarie, "Bifurcation of planar vector fields and Hilbert's sixteenth problem", *Birkhauser-Verlag. Progr. Math. 164*, Basel, 1998.

Order of weak foci and cyclicity

Cyclicity of a singular point

For a given family of polynomial vector fields which is the maximum number of limit cycles that bifurcate from an elementary weak focus or an elementary center?

Problem

Fixed the degree or the family, the number of limit cycles coincides with the order of the weak focus ?

Theorem

For a general system, the number of limit cycles that bifurcate from a weak focus of order K $(V_{2K+1} \neq 0)$ is K.

R. Roussarie, "Bifurcation of planar vector fields and Hilbert's sixteenth problem", *Birkhauser-Verlag. Progr. Math. 164*, Basel, 1998.

Order of weak foci and cyclicity

Cyclicity of a singular point

For a given family of polynomial vector fields which is the maximum number of limit cycles that bifurcate from an elementary weak focus or an elementary center?

Problem

Fixed the degree or the family, the number of limit cycles coincides with the order of the weak focus ?

Theorem

For a general system, the number of limit cycles that bifurcate from a weak focus of order K ($V_{2K+1} \neq 0$) is K.

R. Roussarie, "Bifurcation of planar vector fields and Hilbert's sixteenth problem", *Birkhauser-Verlag. Progr. Math. 164*, Basel, 1998.

Cyclicity is not the number of Lyapunov constants

Theorem (GasGin2010)

Consider a one-parameter family of differential systems of the form

$$\begin{cases} x' = -y + a^k x(x^2+y^2) + aP(x,y,a), \\ y' = -x + a^k y(x^2+y^2) + aQ(x,y,a), \end{cases}$$

where P and Q are analytic functions, starting at least with terms of degree 4 in x and y, and $k \geq 1$ is an integer number. Then:

- *The first Lyapunov constant is $V_3 = 2\pi a^k$ and the origin is a center if and only $a = 0$.*
- *The cyclicity of the origin is at most $k-1$ and there are analytic functions, P and Q, for which this upper bound is sharp.*

A. Gasull & J. Giné. "Cyclicity versus center problem". *Qual. Theory Dyn. Syst.* **9** (2010) 101–111.

$M(n) \leq H(n)$

Definition

- $M(n)$ is the number of small amplitude limit cycles bifurcating from an elementary center or an elementary focus in the class of polynomial vector fields of degree n.
- The Hilbert number $H(n)$ is the maximal number of (all) limit cycles in the class of polynomial vector fields of degree n.

Non elementary centers

If the center is not elementary

Theorem (LiLiLliZha2001)

From the center $(x', y') = (-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x})$ *with* $H = \frac{1}{n+1}x^{n+1} + \frac{1}{2}y^2$ *bifurcate at least* $(n^2 + 4n - 5)/8$ *limit cycles.*

C. Li, W. Li, J. Llibre, Z. Zhang. "Polynomial systems: a lower bound for the weakened 16th Hilbert problem". *Extracta Mathematicae* 16 (2001) 441–447.

$M(2) = 3$ and $M(3) \geq 11$

Proposition (Bau1954)

There are systems of degree 2 with 3 limit cycles surrounding an elementary center or an elementary weak focus.

N. N. Bautin. "On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type". *Amer. Math. Soc. Transl.* 100 (1954)

Proposition (Zol1995, Chr2006)

There are systems of degree 3 with 11 limit cycles surrounding an elementary center or an elementary weak focus.

H. Zoladek. "Eleven small limit cycles in a cubic vector field". *Nonlinearity* 8 (1995), 843–860.

C. Christopher. "Estimating limit cycle bifurcations from centers". *Diff. Eq. with Symbolic Computation* (2006) 23–35.

Linear parts of Lyapunov constants

Cyclicity problem

$K+1$ Lyapunov constants (adding the trace) provide K limit cycles?

Theorem (Chr2006)

Suppose that s is a point on the center variety and that the first k of the Lyapunov constants (V_i) have independent linear parts (with respect to the expansion of V_i about s), then s lies on a component of the center variety of codimension at least k and there are bifurcations which produce $k-1$ limit cycles locally from the center corresponding to the parameter value s.

C. Christopher. "Estimating limit cycle bifurcations from centers". *Diff. Eq. with Symbolic Computation* (2006) 23–35.

Linear parts of Lyapunov constants

Cyclicity problem

$K+1$ Lyapunov constants (adding the trace) provide K limit cycles?

Theorem (Chr2006)

Suppose that s is a point on the center variety and that the first k of the Lyapunov constants (V_i) have independent linear parts (with respect to the expansion of V_i about s), then s lies on a component of the center variety of codimension at least k and there are bifurcations which produce $k-1$ limit cycles locally from the center corresponding to the parameter value s.

C. Christopher. "Estimating limit cycle bifurcations from centers". *Diff. Eq. with Symbolic Computation* (2006) 23–35.

$M(4) \geq 18$

Proposition (HaiTor2015)

The equation of degree 4

$$z' = i\,z + 2i\,z^4 + i\,z\bar{z}^3 + \sqrt{\frac{52278}{20723}}\,\bar{z}^4$$

has 18 limit cycles bifurcating from the origin

The order of weak focus coincides with the cyclicity.

Proof. After a general perturbation of degree 4, the matrix formed by the linear part (with respect to the perturbation parameters) of the first Lyapunov constants has maximal rank. □

H. Liang & J. Torregrosa. "Weak foci of high order and cyclicity". *Preprint* (2015). Submitted.

$M(4) \geq 21$, $M(5) \geq 26$

Theorem (Gin2012)

Let s be a real constant. From the origin of the equation of degree 4

$$z' = iz - \frac{5s^2+1}{2(s+i)^2} z^4 + \frac{3s^2-1-2s\,i}{s^2+1} z^3 w + \frac{s^2-1}{(s-i)^2} z w^3 - \frac{3s^2-1+4s\,i}{2(s-i)^2} w^4$$

bifurcate 21 limit cycles after a perturbation with polynomials of degree 4.

Proof. The system (without perturbation) is a center. (1) The linear parts (with respect to the perturbation parameters) of the first 15 Lyapunov constants are linearly independent. (2) Using the second order terms. (3) Perturbing with linear terms. Hence, there is a one parameter family of weak focus of order 21 with 21 small bifurcated limit cycles. □

J. Giné "Higher order limit cycle bifurcations from non-degenerate centers". *Appl. Math. and Comp.* 218 (2012) 8853–8860.

$M(4) \geq 21$, $M(5) \geq 26$

Theorem (Gin2012)

Let s be a real constant. From the origin of the equation of degree 4

$$z' = iz - \frac{5s^2+1}{2(s+i)^2} z^4 + \frac{3s^2-1-2s\,i}{s^2+1} z^3 w + \frac{s^2-1}{(s-i)^2} z w^3 - \frac{3s^2-1+4s\,i}{2(s-i)^2} w^4$$

bifurcate 21 limit cycles after a perturbation with polynomials of degree 4.

Proof. The system (without perturbation) is a center. (1) The linear parts (with respect to the perturbation parameters) of the first 15 Lyapunov constants are linearly independent. (2) Using the second order terms. (3) Perturbing with linear terms. Hence, there is a one parameter family of weak focus of order 21 with 21 small bifurcated limit cycles. □

J. Giné "Higher order limit cycle bifurcations from non-degenerate centers". *Appl. Math. and Comp.* 218 (2012) 8853–8860.

Perturbing known weak foci and centers

n	2	3	4	5	6	7
n^2+n-2	4	10	18	28	40	54
wf	3	-	18	18	39	34
centers	3	11	21	26	-	-

Theorem

1. *The weak foci of HuaWanWanYan2008 for $n=4$ and $n=5$ of order 18 have cyclicity 18.*
2. *The weak focus of QiuYan2010 for $n=6$ of order 40 has cyclicity at least 39.*
3. *The weak focus of LiaTor2015 for $n=7$ of order 36 has cyclicity at least 34.*

H. Liang & J. Torregrosa. "Weak foci of high order and cyclicity". *Preprint* (2015). Submitted.

Summary: $M(n) \geq ?$

n	2	3	4	5	6	7
n^2+n-2	4	10	18	28	40	54
wf	3	-	18	18	39	34
centers	3	11	21	26	-	-
$M(n) \geq$	3	11	21	26	39	34

Questions

- How can we improve these values?
- How can we find new weak foci of high order?
- For every degree n, which is the best center to perturb?

Linear term of Lyapunov constants: Parallelization

Theorem (LiaTor2015)

Let $p(z,\bar{z})$ *be a polynomial starting with terms of degree 2. Let* $Q_i(z,\bar{z},\boldsymbol{\lambda})$ *be analytic functions such that* $Q_i(0,0,\boldsymbol{\lambda}) \equiv 0$ *and* $Q_i(z,\bar{z},\mathbf{0}) \equiv 0$, *for* $i = 1,\ldots,s$. *Let* $a_1,\ldots,a_s$ *be any s fixed constants. Suppose that* $V_k^{Q_i}$ *are the k-Lyapunov constants of equations*

$$\dot{z} = iz + p(z,\bar{z}) + Q_i(z,\bar{z},\boldsymbol{\lambda}), \ \boldsymbol{\lambda} \in \mathbb{C}^m, \quad \text{for } i = 1,\ldots,s.$$

Then the linear part of $a_1 V_k^{Q_1} + \cdots + a_s V_k^{Q_s}$ *is the linear part of the k-Lyapunov constant of equation*

$$\dot{z} = iz + p(z,\bar{z}) + a_1 Q_1(z,\bar{z},\boldsymbol{\lambda}) + \cdots + a_s Q_s(z,\bar{z},\boldsymbol{\lambda}),$$

with respect to the parameters $\boldsymbol{\lambda}$.

H. Liang & J. Torregrosa. “Parallelization of the computation of Lyapunov constants and cyclicity of centers”. Work in progress.

Perturbing a Darboux quadratic center

The system

$$\begin{aligned}\dot{x} &= -y + 18x^2 + 8xy - 8y^2,\\ \dot{y} &= x + 4x^2 + 14xy - 4y^2,\end{aligned}$$

has a first integral

$$H = \frac{(80x^3 - 480x^2y + 960xy^2 - 640y^3 + 120xy - 240y^2 - 30y - 1)^2}{(20x^2 - 80xy + 80y^2 + 20y + 1)^3}$$

and has a center at the origin.

In complex coordinates

$$\dot{z} = iz + 10z^2 + 5z\bar{z} + (3+4i)\bar{z}^2 + \lambda_2 iz^2 + \lambda_3 z\bar{z} + \lambda_4\bar{z}^2,$$

where $\lambda_2, \lambda_3, \lambda_4$ are real parameters.

Perturbing a Darboux quadratic center

The system

$$\begin{aligned}\dot{x} &= -y + 18x^2 + 8xy - 8y^2,\\ \dot{y} &= \ \ x + 4x^2 + 14xy - 4y^2,\end{aligned}$$

has a first integral

$$H = \frac{(80x^3 - 480x^2y + 960xy^2 - 640y^3 + 120xy - 240y^2 - 30y - 1)^2}{(20x^2 - 80xy + 80y^2 + 20y + 1)^3}$$

and has a center at the origin.
In complex coordinates

$$\dot{z} = iz + 10z^2 + 5z\bar{z} + (3 + 4i)\bar{z}^2 + \lambda_2 iz^2 + \lambda_3 z\bar{z} + \lambda_4\bar{z}^2,$$

where $\lambda_2, \lambda_3, \lambda_4$ are real parameters.

Computing in parallel (perturbing each center separately)

Computing case by case $(\{\lambda_2 \neq 0,\ \lambda_3 = 0,\ \lambda_4 = 0\},\ldots)$

$$
\begin{aligned}
V_3^{\ell,Q_1} &= -\,10\pi\lambda_2, & V_5^{\ell,Q_1} &= 16000\pi\lambda_2, & V_7^{\ell,Q_1} &= -\,\frac{682934375\pi\lambda_2}{18},\\
V_3^{\ell,Q_2} &= 0, & V_5^{\ell,Q_2} &= \frac{2000\pi\lambda_3}{3}, & V_7^{\ell,Q_2} &= -\,\frac{16356250\pi\lambda_3}{9},\\
V_3^{\ell,Q_3} &= 0, & V_5^{\ell,Q_3} &= 0, & V_7^{\ell,Q_3} &= 18750\pi\lambda_4.
\end{aligned}
$$

Using the above result

$$
\begin{aligned}
V_3^{\ell} &= V_3^{\ell,Q_1} + V_3^{\ell,Q_2} + V_3^{\ell,Q_3} = -\,10\pi\lambda_2,\\
V_5^{\ell} &= V_5^{\ell,Q_1} + V_5^{\ell,Q_2} + V_5^{\ell,Q_3} = 16000\pi\lambda_2 + \frac{2000\pi\lambda_3}{3},\\
V_7^{\ell} &= V_7^{\ell,Q_1} + V_7^{\ell,Q_2} + V_7^{\ell,Q_3} = -\,\frac{682934375\pi\lambda_2}{18} - \frac{16356250\pi\lambda_3}{9} + 18750\pi\lambda_4.
\end{aligned}
$$

Computing in parallel (perturbing each center separately)

Computing case by case $(\{\lambda_2 \neq 0,\ \lambda_3 = 0,\ \lambda_4 = 0\},\ldots)$

$$
\begin{aligned}
V_3^{\ell,Q_1} &= -10\pi\lambda_2, & V_5^{\ell,Q_1} &= 16000\pi\lambda_2, & V_7^{\ell,Q_1} &= -\frac{682934375\pi\lambda_2}{18},\\
V_3^{\ell,Q_2} &= 0, & V_5^{\ell,Q_2} &= \frac{2000\pi\lambda_3}{3}, & V_7^{\ell,Q_2} &= -\frac{16356250\pi\lambda_3}{9},\\
V_3^{\ell,Q_3} &= 0, & V_5^{\ell,Q_3} &= 0, & V_7^{\ell,Q_3} &= 18750\pi\lambda_4.
\end{aligned}
$$

Using the above result

$$
\begin{aligned}
V_3^{\ell} &= V_3^{\ell,Q_1} + V_3^{\ell,Q_2} + V_3^{\ell,Q_3} = -10\pi\lambda_2,\\
V_5^{\ell} &= V_5^{\ell,Q_1} + V_5^{\ell,Q_2} + V_5^{\ell,Q_3} = 16000\pi\lambda_2 + \frac{2000\pi\lambda_3}{3},\\
V_7^{\ell} &= V_7^{\ell,Q_1} + V_7^{\ell,Q_2} + V_7^{\ell,Q_3} = -\frac{682934375\pi\lambda_2}{18} - \frac{16356250\pi\lambda_3}{9} + 18750\pi\lambda_4.
\end{aligned}
$$

The complete Lyapunov constants

$$
\begin{aligned}
V_3 =& -2\pi(5+\lambda_3)\lambda_2,\\
V_5 =& \frac{2\pi}{3}(5+\lambda_3)(4800\lambda_2+200\lambda_3+8\lambda_2^2+759\lambda_2\lambda_3-25\lambda_2\lambda_4+8\lambda_3^2\\
&+18\lambda_2^3+27\lambda_2\lambda_3^2+3\lambda_2\lambda_3\lambda_4),\\
V_7 =& -\frac{\pi}{18}(5+\lambda_3)(136586875\lambda_2+6542500\lambda_3-67500\lambda_4-876\lambda_3^2\lambda_4\\
&-732150\lambda_2^2+41353200\lambda_2\lambda_3-491150\lambda_2\lambda_4+1216450\lambda_3^2\\
&-24600\lambda_3\lambda_4+938100\lambda_2^3+31300\lambda_2^2\lambda_3-336\lambda_2^2\lambda_4\\
&+4525685\lambda_2\lambda_3^2-35558\lambda_2\lambda_3\lambda_4+69240\lambda_3^3+3816\lambda_2^4\\
&+150504\lambda_2^3\lambda_3-28592\lambda_2^3\lambda_4+2082\lambda_2^2\lambda_3^2+209256\lambda_2\lambda_3^3\\
&+15408\lambda_2\lambda_3^2\lambda_4+1242\lambda_3^4+1944\lambda_2^5+4572\lambda_2^3\lambda_3^2+8\lambda_2^3\lambda_3\lambda_4\\
&+3384\lambda_2\lambda_3^4+1112\lambda_2\lambda_3^3\lambda_4).
\end{aligned}
$$

Perturbing (first order) Polynomial Hamiltonian

n	2	3	4	5	6	7	
	4	10	18	28	40	54	n^2+n-2
wf	3	11	21	26	39	34	$n^2+n-2?$
Ham	2	5	9	14	20	27	$(n^2+n-2)/2$
			4s (72s)	21s (8.3m)	1.6m (0.9h)	6.6m (4.6h)	Time 1 case
			1000	1000	300	50	Number of random cases

$$
\begin{aligned}
x' &= -\frac{\partial H}{\partial y} + \mathcal{E}_P(x,y),\\
y' &= \frac{\partial H}{\partial x} + \mathcal{E}_Q(x,y).
\end{aligned}
$$

$$H(x,y) = \frac{1}{2}(x^2+y^2) + H_3 + H_4 + \ldots + H_{n+1}$$

Perturbing (first order) Reversible Centers

n	2	3	4	5	6	7	n
	4	10	18	28	40	54	n^2+n-2
wf	3	11	21	26	39	34	n^2+n-2 ?
Ham	2	5	9	14	20	27	$(n^2+n-2)/2$
Rev	2	6	11 3s (46s)	17 11s (4.4m)	24 41s (23m)	32 2.2m (1.6h)	$(n^2+3n-6)/2$ Time 1 case
			250	250	250	250	Number of random cases

$$\begin{aligned} x' &= -y + p(x,y) + \mathcal{E}_P(x,y), \\ y' &= x + q(x,y) + \mathcal{E}_Q(x,y) \end{aligned}$$

with $p(-x,y) = p(x,y)$ and $p(-x,y) = -q(x,y)$.

Perturbing (first order) Rational Darboux Centers

n	2	3	4	5	6	7	n
	4	10	21	28	40	54	n^2+n-2
wf	3	11	21	26	39	34	n^2+n-2 ?
Ham	2	5	9	14	20	27	$(n^2+n-2)/2$
Rev	2	6	11	17	24	32	$(n^2+3n-6)/2$
Darboux		10	16 24s (6m)	26 4m (1.6h)	35 20m (11.3h)	47 3.6h (7d)	Time 1 case

$$H = \frac{(x\,y^2 + A\,x + B)^{n+2}}{x^n(x\,p(y)+q(y))^2}$$

(Inspired in Zoladek's examples)

with $p(y) = \sum_{i=0}^{n-4} a_i y^i + \frac{n(n+2)A^2}{8} y^{n-2} + \frac{(n+2)A}{2} y^n + y^{n+2}$ and

$$q(y) = \sum_{i=0}^{n-3} b_i y^i + \frac{n(n+2)AB}{4} y^{n-2} + \frac{(n+2)B}{2} y^n$$

Perturbing different families of centers

n	2	3	4	5	6	7	n
	4	10	18	28	40	54	n^2+n-2
wf	3	11	21	26	39	34	n^2+n-2 ?
Ham	2	5	9	14	20	27	$(n^2+n-2)/2$
Rev	2	6	11	17	24	32	$(n^2+3n-6)/2$
Darboux	**3**	**11**	**21**	26	35	47	n^2+3n-7^* ?

The number of parameters is $(n^2+3n-4)+1$.

* Conjectured by J. Giné.

J. Giné "Higher order limit cycle bifurcations from non-degenerate centers". *Appl. Math. and Comp.* 218 (2012) 8853–8860.

Perturbing Holomorphic Centers

n	2	3	4	5	6	7	n
	4	10	18	28	40	54	n^2+n-2
wf	3	11	21	26	39	34	n^2+n-2 ?
Ham	2	5	9	14	20	27	$(n^2+n-2)/2$
Rev	2	6	11	17	24	32	$(n^2+3n-6)/2$
Darboux	**3**	**11**	**21**	26	35	47	n^2+3n-7?
Hol	2	9	18 7s (1.7m)	**28** 33s (12.2m)	**40** 2m (1.2h)	**54** 8m (5.8h)	n^2+n-2 Time 1 case
			50	50	50	50	Number of random cases

$$z' = iz + f(z) + \mathcal{E}(z, \bar{z}),$$

$M(n) \geq n^2 + n - 2$ for $4 \leq n \leq 12$

Theorem (LiaTor2015)

For $4 \leq n \leq 12$, equation

$$\dot{z} = iz + z^2 + z^3 + \cdots + z^n + \lambda_1 z + \sum_{k+\ell=2}^{n} \lambda_{k,\ell} z^k \bar{z}^\ell,$$

where $\lambda_1 \in \mathbb{R}$, $\lambda_{k,\ell} \in \mathbb{C}$ are perturbing parameters, has at least $n^2 + n - 2$ small limit cycles bifurcating from the origin.

H. Liang & J. Torregrosa. "Parallelization of the computation of Lyapunov constants and cyclicity of centers". Work in progress.

$M(n) \geq n^2 + n - 2$ for $4 \leq n \leq 12$

Proof. For every $4 \leq n \leq 12$ consider the perturbed equation

$$\dot{z} = iz + z^2 + z^3 + \cdots + z^n + \lambda_1 z + \sum_{k+\ell=2}^{n} \lambda_{k,\ell} z^k \bar{z}^\ell,$$

1. The number of total parameters is $(n^2 + 3n - 4) + 1$.
2. First assume $\lambda_1 = 0$.
3. For $\lambda_{k,0} z^k$ terms all the Lyapunov constants are zero.
4. For the remaining $N = n^2 + n - 2$ parameters we compute the linear part of the first N Lyapunov constants.

n	4	5	6	7	8	9	10	11	12
	1.7m	12.2m	1.2h	5.8h	1.4d	4.9d	1.8W	1.1M	3.3M
P64	7s	0.5m	2m	8m	1.1h	3.1h	6.3h	0.9d	2.5d

5. The rank of the matrix $N \times N$ formed by these N linear parts with respecte to the N parameters is N (without λ_1).
6. There is a one parameter family of weak foci of order N that, adding λ_1, have cyclicity N.

$M(n) \geq ?$

Lower bounds for $M(n)$

The number of small amplitude limit cycles bifurcating from an elementary center or an elementary focus in the class of polynomial vector fields of degree n is

- $M(n) \geq n^2 + 3n - 7$ for $n = 2, 3, 4.$
  [Bau1954,Zol1995,Chr2006, Gin2012]
- $M(n) \geq n^2 + n - 2$ for $n = 5, 6, \ldots, 12.$
  [LiaTor2015]

$M(n)$ versus $H(n)$

Corollary

$H(6) \geq M(6) \geq 40$

n	2	3	4	5	6	7
$M(n)$	3	11	21	28	**40**	54
$H(n)$	4	13	22	28	**40**	65

H. Liang & J. Torregrosa. "Parallelization of the computation of Lyapunov constants and cyclicity of centers". Work in progress.

Remark: The simultaneos bifurcation techniques for symmetric centers provide good lower bounds for $H(n)$ only for $n \geq 7$.