DESCRIPTION OF TWO-DIMENSIONAL ATTRACTORS OF SOME DISSIPATIVE INFINITE-DIMENSIONAL DYNAMICAL SYSTEMS

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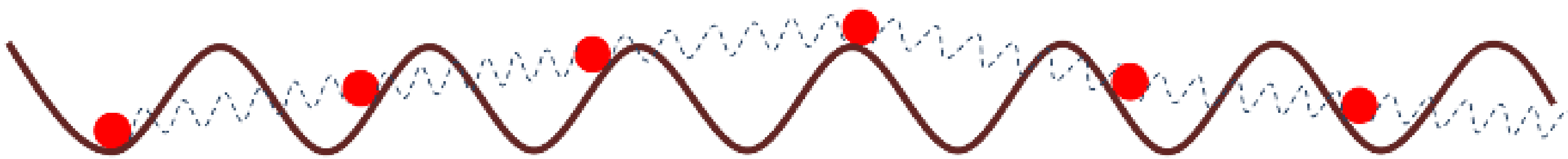
OVERVIEW OF THE TALK

Dissipative dynamics of pulled (driven) Frenkel-Kontorovora models:



Applied AC or DC force F(t)

1. Description of the model and known numerical observations

2. The (at most) 2D representation of the attractor – theory and numerics

3. Description of phase transitions and asymptotics

4. Applications: new numerical algorithms to determine phase transitions

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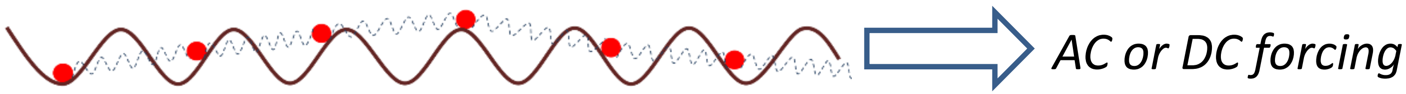
1 THE THEORY APPLIES TO 1D, DRIVEN DISSIPATIVE DYNAMICS

Overdamped, driven FK dynamics:



$$
\begin{aligned}
H(x) &= \sum_{n=-\infty}^{\infty} \left(W(x_n - x_{n+1}) + V(x_n)\right) \\
\frac{dx_n}{dt} &= -\frac{\partial H(x)}{\partial x_n} + F(t) \\
&= W'(x_{n+1} - x_n) - W'(x_n - x_{n-1}) - V'(x_n) + F(t).
\end{aligned}
\qquad \text{(FK)}
$$

- *F(t)* constant - DC dynamics
- *F(t)* time-periodic - AC dynamics
- *V(x)* – on-site (periodic) potential, e.g. $V(x) = k\sin(2\pi x)$
- W(x) – interaction potential, e.g. $W(x-y) = (x-y)^2/2$

1 OTHER MODELS TO WHICH THE THEORY APPLIES

The theory also applies to (not included in this talk) :

- Damped FK dynamics, with sufficiently strong damping:

$$\frac{d^2 x_n}{dt^2} + \gamma \frac{dx_n}{dt} = -\frac{\partial H(x)}{\partial x_n} + F(t)$$

- Reaction-diffusion in 1D (f periodic in t,x):

$$u_t = u_{xx} + f(t, x, u, u_x)$$

- In particular, Burger's equation:

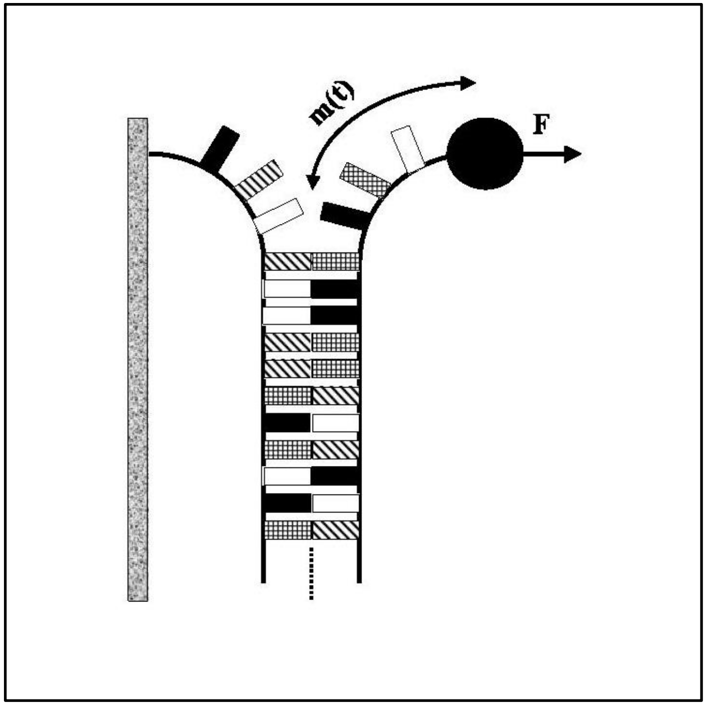
$$u_t = u_{xx} - u u_x$$

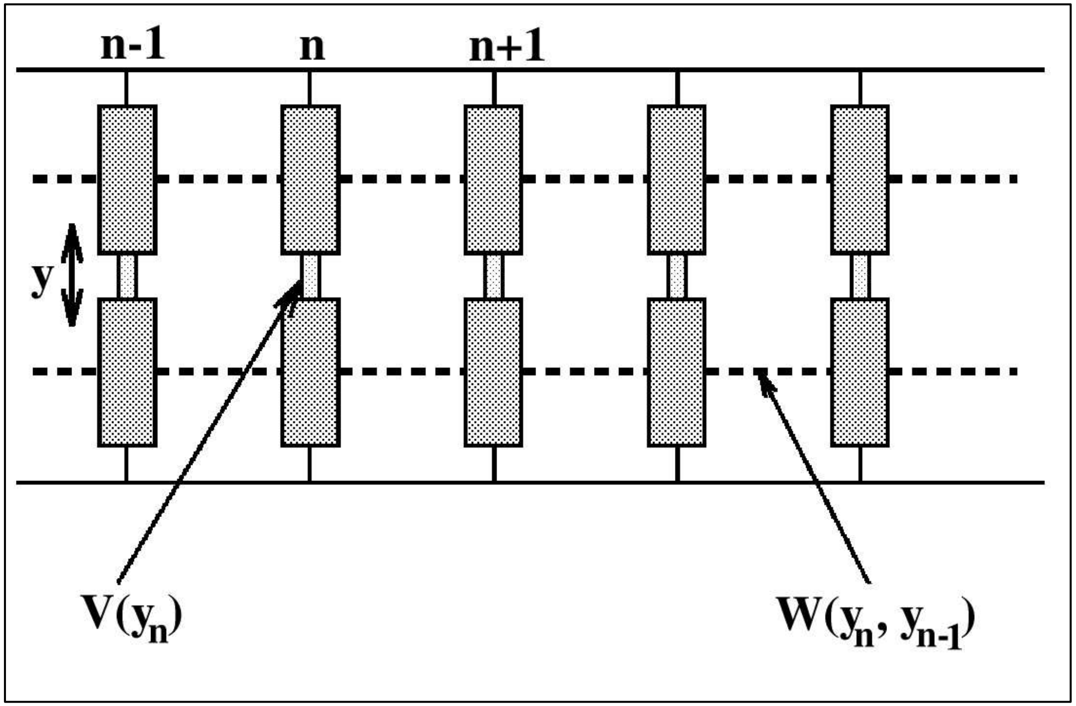
- Damped hyperbolic equation in 1D with sufficiently strong damping:

$$u_{tt} + \gamma u_t = u_{xx} + f(t, x, u, u_x)$$

1 EXAMPLES OF PHYSICAL MODELS

- **DNA unzipping** (in replication and transcription)
- Peyrard – Bishop – Dauxois model (Floria, Baesens, Gomez-Gardenez, 2006)
- V – interaction between nucleotides; W – interaction between neighbouring pairs; F – unzipping force; y – the distance between nucleotide pairs





- **Other physical models**: Charge density wave transport; Josephson junction arrays; dislocation dynamics in solids; in surface physics ...

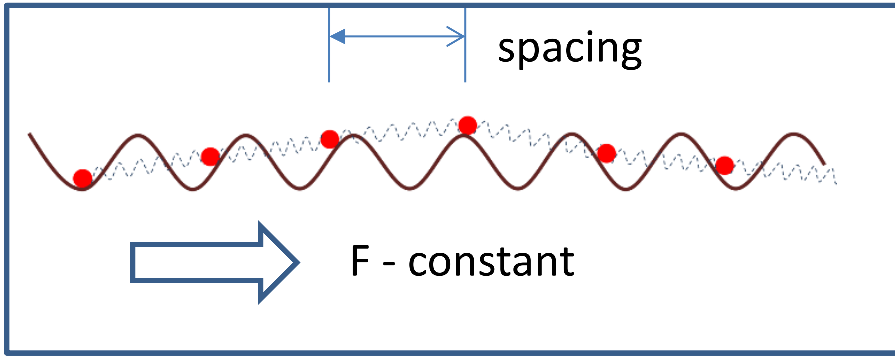
1 NUMERICALLY OBSERVED BEHAVIOR – DC DRIVING

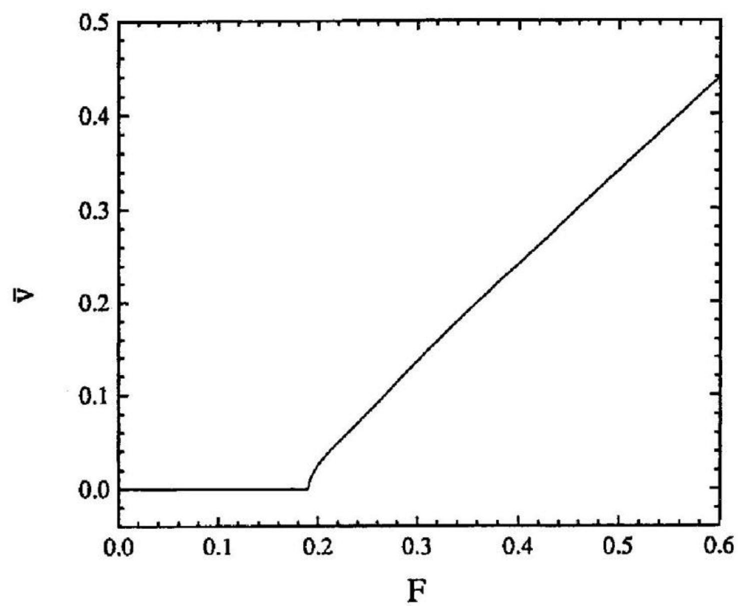
Rigorously known:

- „Depinning" („unlocking") force typically non-zero
- Depinning force and sliding speed depend on the mean spacing
- Non-zero speed for mean spacing *r*: there exists unique „ordered" orbit (uniformly sliding state)

No / limited rigorous results:

- Sharp estimates of depinning force?
- Asymptotics for various initial conditions (covergence to the sliding solution?)
- Behavior close to pinning/depinning (unlocking) transition?
- Speed of convergence?





Typical dependence of speed on force (source: Floria, Mazo, 1996)

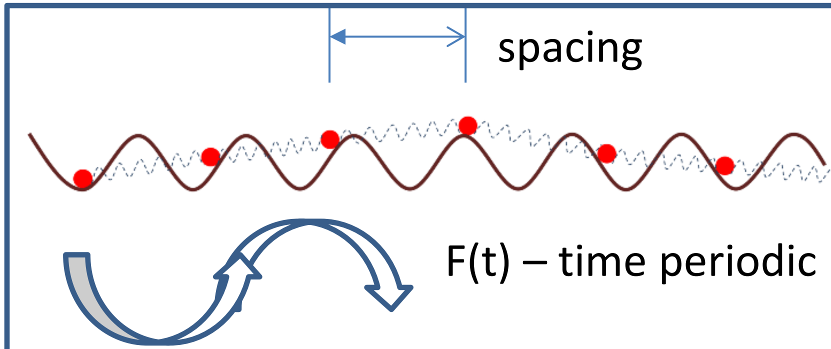
1 NUMERICALLY OBSERVED BEHAVIOR – AC DRIVING

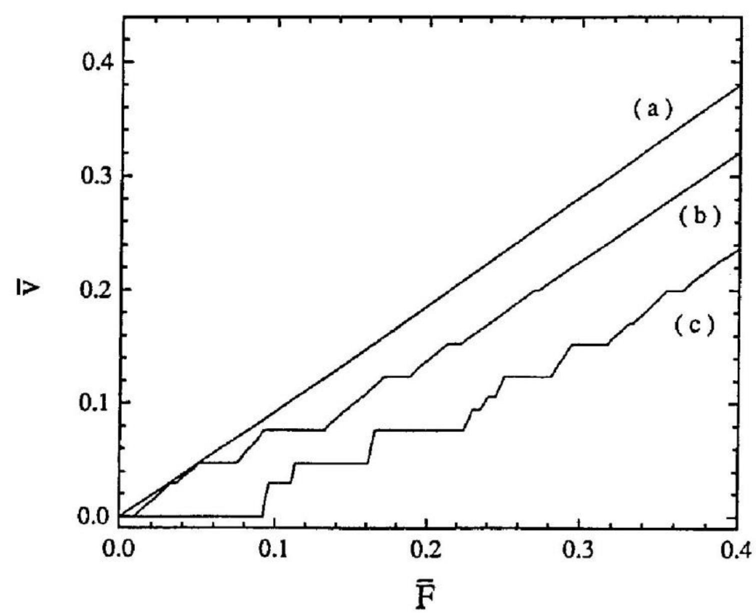
Rigorously known:

- There exist ordered (synchronized) orbits for any mean spacing (Qin, 2013)

No / limited rigorous results:

- Rigorous explanation of „mode locking”?
- Is the v(F) dependence a devil staircase?
- Description of the „dynamical Aubry transition”?
- Convergence to synchronized orbits / asymptotics for arbitrary initial conditions?
- Speed of convergence?
- Etc.

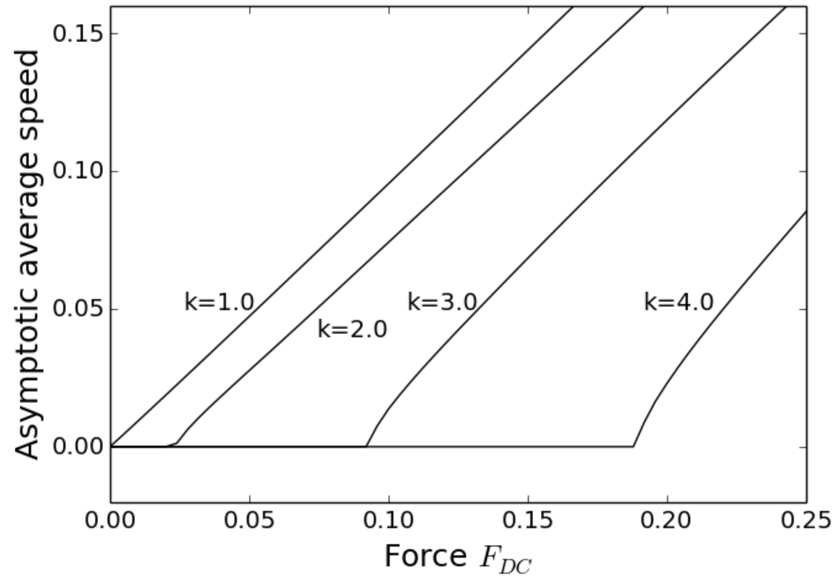




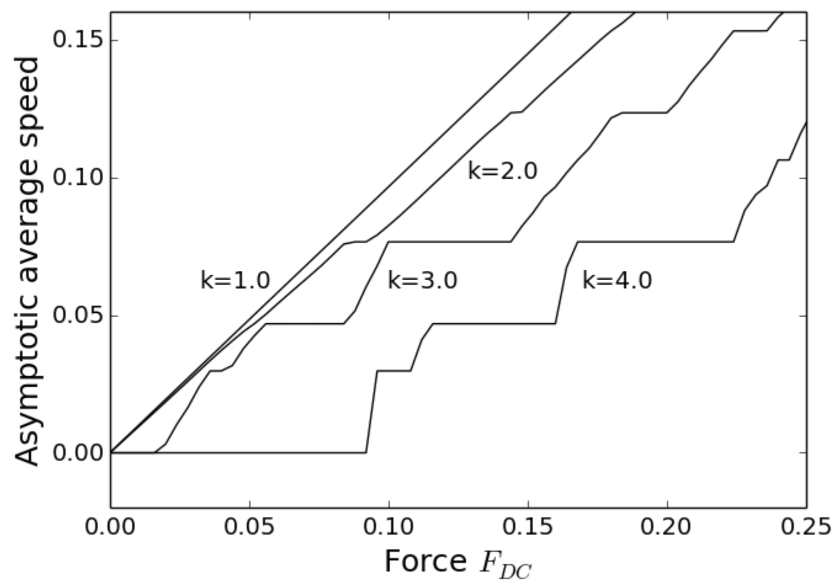
Typical dependence of speed on force for three FK models - different site potentials (source: Floria, Mazo, 1996)

1 MORE NUMERICS : AVERAGE SPEED VS. AVERAGE FORCE

DC driving:



AC driving:



2 THE MATHEMATICAL BACKGROUND

The Aubry-Mather theory

- Representation of ground states of FK model as a twist map
- Commensurations / discomensurations

Poincare-Bendixson theorem for 1D reaction-diffusion equations

- Fiedler, Mallet-Paret, 1989
- Asymptotics for reaction-diffusion on bounded domains

Ergodic theory

- SRB measures
- Physical measures
- Minimising measures

Hamiltonian dyn.

- KAM theory
- Break-up of invariant tori (Converse KAM)
- Renormalization theory

2 THE AUBRY – MATHER THEORY

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Description of equilibria:

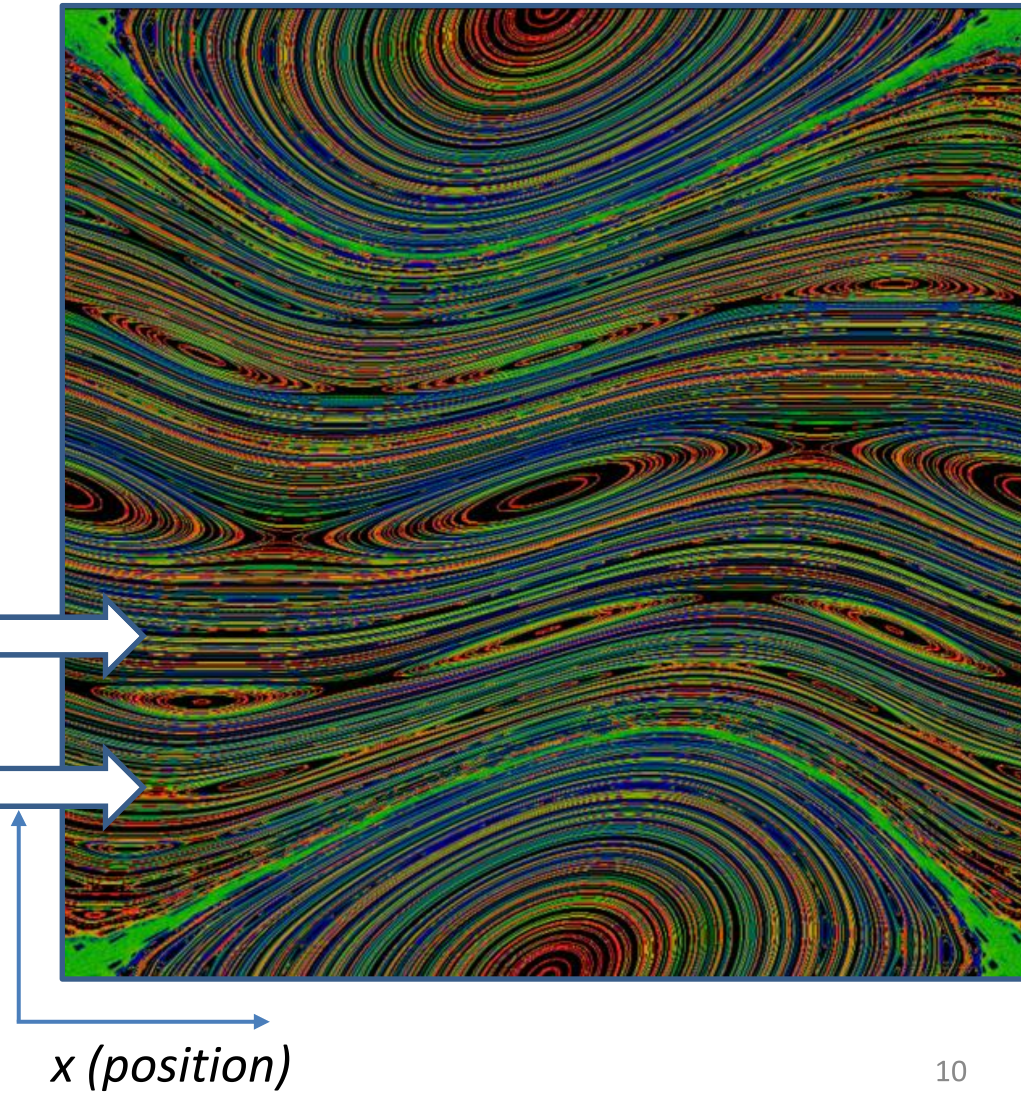
- Elementary: all equilibria (F=0, du/dt=0) characterized as orbits of a 2D twist area-preserving map
- **Aubry-Mather**: existence of ground states for arbitrary mean spacing (=Aubry-Mather sets)
- Ground states are ordered
- Ground states lie on either

Invariant torus (circle)

or

Cantor set

Phase portrait of the 2D representation:



2 POINCARE-BENDIXSON THEOREM FOR REACTION-DIFFUSION EQUATIONS

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- **Equation**: reaction-diffusion in 1d on [0,1], periodic boundary conditions

$$
\begin{aligned}
u_t &= u_{xx} + f(x, u, u_x) \\
u(0,t) &= u(1,t) \\
u(.,0) &= u^0(x)
\end{aligned}
$$

- **Theorem** (Fiedler, Mallet-Paret, 1989): The ω-limit set $\omega(u)$ for any u projects injectively to a compact 2D set
- **Similar theorem for FK model** (Baesens, MacKay, 1998): For finite FK model with periodic boundary conditions
- **Key insight**: the „intersection-counting” („lap-number”) function is a discrete Lyapunov function

2 PHYSICAL SPACE-TIME MEASURES

- Physical (probability, invariant) measures: time averages of any observable on the basin of attraction converge to the spatial average
- **Known** for uniformly hyperbolic, Axiom A systems: unique physical measure (SRB measure)
- Adapted definition to our setting
- Let $K \subseteq \boldsymbol{R}^{\boldsymbol{Z}}$ be a (compact) set of FK chain configurations $(u_n)_{n \in \boldsymbol{Z}}$

Definition: We say that a time- and space-invariant (probability) measure μ on K is *space-time physical*, if for any $u^0 \in K$, and any cont. function *f* on K,

$$\underbrace{\lim_{\boldsymbol{n}\to\infty} \lim_{T\to\infty} \frac{1}{2nT} \int_0^T \sum_{m=-n}^{n} f(S^m u(t))dt}_{\text{Space-time average}} = \underbrace{\int f(u)d\mu(u)}_{\text{Expectation w.r. to physical measure}}$$

Space-time average = Expectation w.r. to physical measure

2 THE MAIN RESULT: ATTRACTOR IS (AT MOST) 2D

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Theorem (S.Sl., 2014): The attractor *A* for AC and DC dissipatively driven FK model is at most 2-dimensional.

The injective projection is given with $\pi: A \to R^2$,

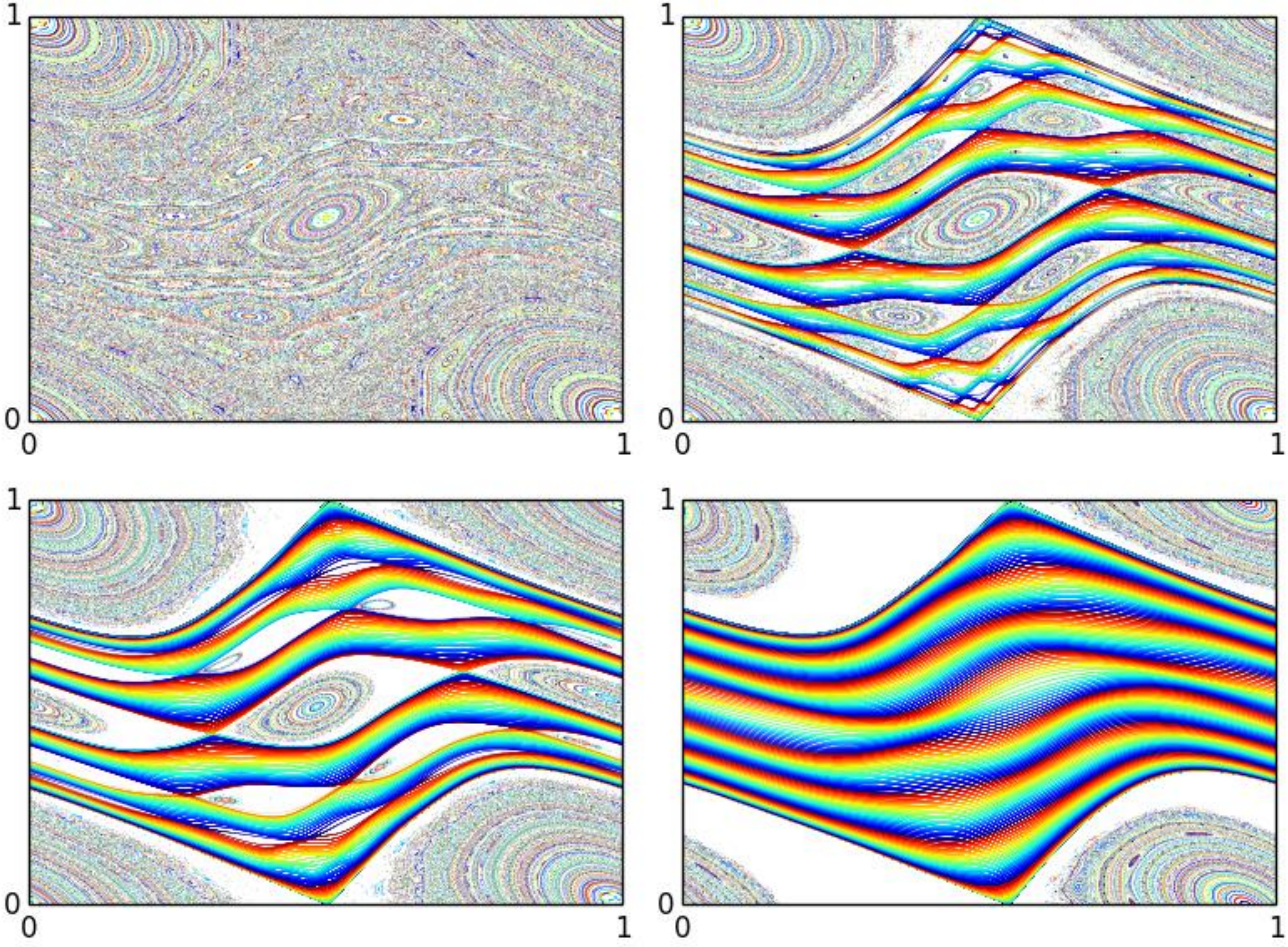
$$\pi\big((x_n)\big) = (x_0, x_1 - x_0)$$

Definition of the attractor – in an ergodic theoretical sense.

Equivalent definitions of the attractor

- Configurations „observable" for positive density of times and spatial translates,
- Union of supports of all space-time invariant measures,
- Configurations „observable" for positive space-time probability.

2 NUMERICS: 2D REPRESENTATIONS OF THE ATTRACTOR



2D representations of the attractor of a DC-driven standard FK model, with k=1.0. The DC force (left to right): F=0, 0.001, 0.005, 0.05. The same color corresponds to the same configuration and its time evolution.

3 WHAT IS A DYNAMICAL PHASE TRANSITION?

Confusion in the literature: how to recognize dynamical (Aubry) phase transition?

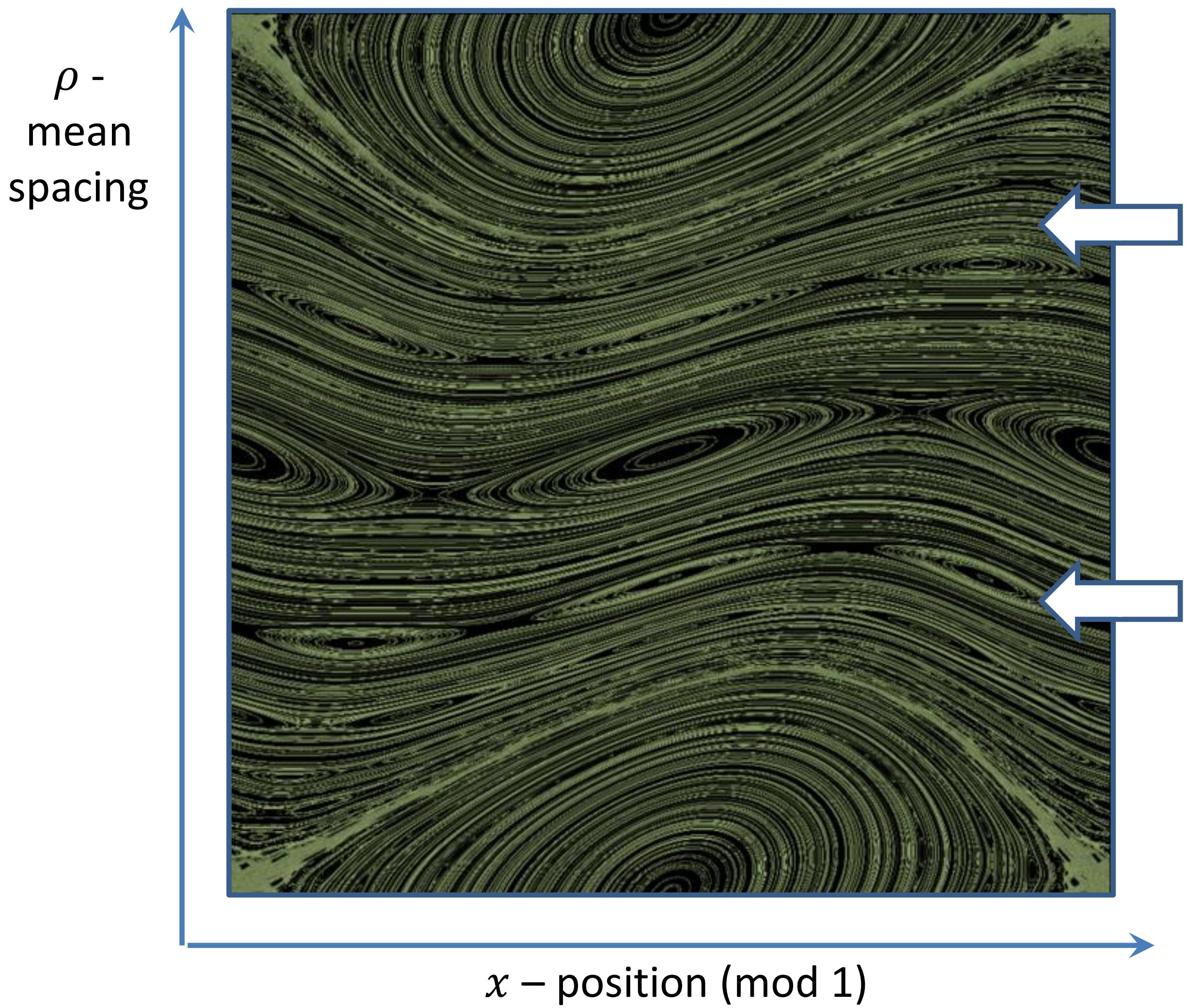
- DC case „clear”: when the chain starts moving
- AC case – unclear, speed vs. force dependency complex
- We distinguish pinned vs. depinned phase (or locked vs. unlocked)
- Pinned phase: part of the physical space asymptotically „off-limit”

Theorem (S.Sl., 2014): The following characterizations of the depinned phase (for fixed mean spacing) are equivalent:

- Projection of the attractor to the first coordinate covers the entire real line (in the pinned phase, it is a Cantor set)
- The space-time invariant measure is unique
- The modulation function is smooth (in the pinned phase, it is a Devil’s staircase)

3 EXAMPLE – DC DRIVING (SIMILAR PICTURE IN THE AC CASE!!)

Constant driving force *F,* 2D representation of the attractor



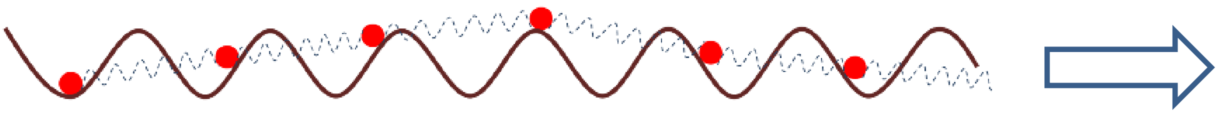
Invariant circle at level ρ:

- Depinned phase
- Not zero average speed
- Unique solution in the attractor

No invariant circle at level ρ:

- Pinned phase
- Zero average speed
- Many metastable states
- Dynamics depends on initial conditions

3 ERGODIC THEORETICAL INTERPRETATION OF RESULTS



AC or DC force F(t):

{ Union of supports of physical* measures }	⊂ { Union of supports of invariant* measures }	⊂ { Classical attractor }
• Asymptotics for a.e. time and a.e. initial condition	• Asymptotics for a.e. time and a.e. initial condition	• Asymptotics for all times and all initial conditions
• Circles + cantori	• 2D set	• ∞-dim. set
• Generalized Aubry-Mather sets	• Characterization as orbits of a 2D twist-like map	• Description hopeless (?)
• Consists of synchronized orbits		

new results

* space-time physical / space-time invariant

3 WHAT ARE SYNCHRONIZED ORBITS?

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Definition: Let *u(t)* be an orbit of (FK). We say that *u(t)* is synchronized, if the set $\{S^n u(t),\ t \in \boldsymbol{R}, n \in \boldsymbol{Z}\}$ is totally ordered.

- Here $(S^n u(t))_m = u(t)_{n+m}$ is the spatial shift
- Two configurations totally ordered = their graphs do not intersect

Theorem: The equation (FK) in both AC and DC cases for each mean spacing $\rho \in \boldsymbol{R}$ has a synchronized solution.

- In the DC case by Middleton (1992), Baesens, MacKay (1998), Qin (2010, 2011)
- In the AC case Hu, Qin, Zheng (2005), Qin, S. Sl. (2013)

3 SYNCHRONIZED ORBITS ARE ATTRACTING

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Theorem: (S.Sl., 2014) In both AC and DC cases, **depinned phase** (for fixed mean spacing $\rho \in \boldsymbol{R}$:

- ω-limit set for any initial condition* with mean spacing $\rho \in \boldsymbol{R}$ consists of synchronized solution

Theorem: (S.Sl., 2014) In both AC and DC cases, **pinned phase** (for fixed mean spacing $\rho \in \boldsymbol{R}$ is locally stable.

* asymptotics defined in ergodic-theoretical sense (orbits in the closure observable for positive density of times and spatial translates)

Complete description of the asymptotics: 2D dynamics as above + coarsening (see e.g. Eckmann, Rougemont; dynamics of the real Ginzburg-Landau equation)

4 APPLICATIONS

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Problem	New tool available
• DC: sharp estimate of the unlocking transition • AC: sharp estimate of the dynamical Aubry transition	• Criteria for break-up of invariant tori (Boyland, MacKay, Stark) – „Converse KAM”
• AC, DC: persistence of the sliding regime for (sufficiently) irrational mean spacing	• KAM theory
• AC, DC: Behavior close to the pinning/depinning and dynamical Aubry transition	• Renormalization theory approach developed for twist area-preserving maps (?)
• AC, DC: Dependence of speed on parameters	• Various ergodic-theoretical tools
• Speed of convergence to synchronized solutions	• Further study of the key tool – new Lyapunov functions on the space of measures

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THANK YOU