Classifications of parabolic germs and ε-neighborhoods of orbits

Maja Resman

University of Zagreb, Croatia

Theoretical and computational methods in dynamical systems and fractal geometry, Maribor

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Thesis supervisors:
Pavao Mardešić, Université de Bourgogne, France
Vesna Županović, University of Zagreb, Croatia

The objects under consideration

Germs of analytic diffeomorphisms $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$

$$f(z) = \lambda z + a_1 z^{k+1} + a_2 z^{k+2} + \dots,\ \lambda,\ a_i \in \mathbb{C}.$$

- $|\lambda| \neq 1$ hyperbolic case, analytically linearizable, uninteresting
- $\lambda = e^{2\pi i \alpha},\ \alpha \neq \mathbb{Q}$ irrational rotation, very complicated
- $\lambda = e^{2\pi i \alpha},\ \alpha \in \mathbb{Q}$ rational rotation; suppose $\lambda = 1$, PARABOLIC CASE

$\star$ $k+1$... the *multiplicity* of fixed point 0

Local discrete dynamics at the origin

⋆ **Leau-Fatou flower theorem** (1987):

- k attracting directions: $(-a_1)^{-\frac{1}{k}}$; k repelling directions: $a_1^{-\frac{1}{k}}$

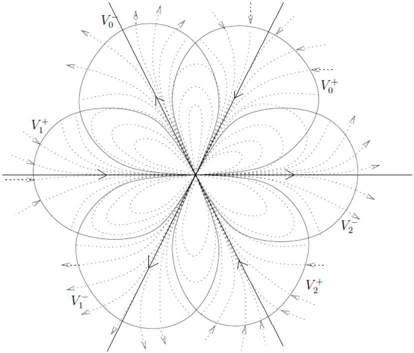


Figure: $f(z) = z + z^4 + o(z^4)$

The problem considered

$O^f(z_0) = \{f^{\circ n}(z_0) : n \in \mathbb{N}_0\}$ ... the orbit of f, initial point z_0

Can we recognize a germ using fractal properties of only one orbit?

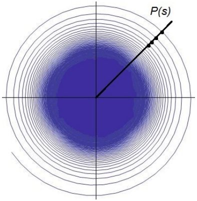
- formal class of a germ
- analytic class of a germ

Motivation for using box dimension: dynamical systems

Box dimension of spiral trajectory locally around singular point reveals complexity in bifurcations of singular point!

- polynomial vector field, focus point at the origin

$$\begin{cases} \dot{x} &= -y + p(x,y), \\ \dot{y} &= x + q(x,y). \end{cases}$$



- The Poincaré map $P(s) = s - s^{2k+1} + o(s^{2k+1})$, $k \in \mathbb{N}_0$; focus of order k
- cyclicity in generic bifurcations: k
- $\dim_B(S(x_0)) = \frac{4k}{2k+1}$

(Žubrinić, Županović, *Fractal analysis of spiral trajectories of some planar vector fields* (2005))

...complex saddles...

Fractal properties of a set $U \subset \mathbb{C}$ $(\mathbb{R}^2)$

THE BOX DIMENSION OF A SET (fractal dimension)

- $U \subset \mathbb{R}^2$ bounded
- $\varepsilon > 0$, $|U_\varepsilon|$ the area of the ε-neighborhood
- For $s \in [0, 2]$, we consider

$$\lim_{\varepsilon \to 0} \frac{|U_\varepsilon|}{\varepsilon^{N-s}} \in [0, \infty],$$

and draw:

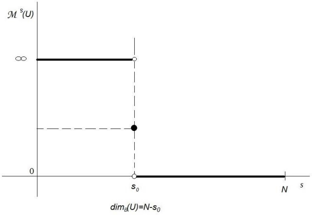


Figure: $s \mapsto \lim_{\varepsilon \to 0} \frac{|U_\varepsilon|}{\varepsilon^{2-s}}$, $s \in [0, 2]$.

- the moment of jump $s_0 \equiv$ the box dimension, $\dim_B(U) = s_0$.
- the value at $s_0 \equiv$ Minkowski content, $\mathcal{M}(U)$.
- if $|U_\varepsilon| \sim C\varepsilon^{2-s} \;\Rightarrow\; \dim_B(U) = s,\; \mathcal{M}(U) = C$.

$\star$ More generally, the complete function

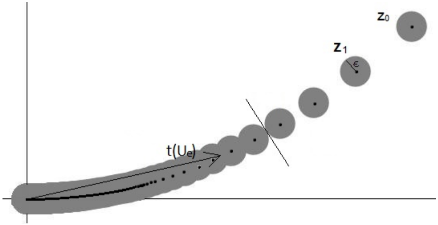
$$\varepsilon \mapsto |U_\varepsilon|$$

as *fractal* property of set U

Definition (The DIRECTED area of the ε-neighborhood, R)

$$A^{\mathbb{C}}(U_\varepsilon) = |U_\varepsilon| \cdot t(U_\varepsilon) \in \mathbb{C},$$

$t(U_\varepsilon) \in \mathbb{C}$ the *center of mass* of U_ε.



Formal classification of parabolic diffeomorphisms

(Birkhoff, Ècalle, Kimura, ~ 1950)

$\star$ formal changes of variables:

1. $\phi_1(z) = c_1 z,$
2. $\phi_i(z) = z + c_i z^i,\ c_i \in \mathbb{C}, \quad i = 2, 3, \ldots$

$$\widehat{\phi}(z) = \ldots \circ \phi_2^{-1} \circ \phi_1^{-1}(z) = \sum_{l=1}^{\infty} d_k z^k \quad \text{(formal series)}$$

$\Rightarrow$ **formal normal form**

$$f_0(z) = \widehat{\phi} \circ f \circ \widehat{\phi}^{-1}(z) = Exp\left(\frac{z^{k+1}}{1 + \frac{\lambda}{2\pi i} z^k} \frac{d}{dz}\right), \quad \lambda \in \mathbb{C},$$

$$\widetilde{f_0}(z) = z + z^{k+1} + \left(\frac{k+1}{2} - \frac{\lambda}{2\pi i}\right) z^{2k+1}.$$

$\star$ formal type: (k, λ), $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$.

Asymptotic expansion of the directed area of the ε-neighborhood of an orbit

$z_0 \in V_+$ (attracting petal);

Theorem (R, 2013)

$$A^{\mathbb{C}}(\varepsilon, z_0) = K_1\varepsilon^{1+\frac{2}{k+1}} + K_2\varepsilon^{1+\frac{3}{k+1}} + \ldots + K_{k-1}\varepsilon^{1+\frac{k}{k+1}} + \\ + H^f(z_0)\varepsilon^2 + K_k\varepsilon^{2+\frac{1}{k+1}}\log\varepsilon + R(z_0,\varepsilon), \qquad (1)$$

$$R(z_0,\varepsilon) = o(\varepsilon^{2+\frac{1}{k+1}}\log\varepsilon),\ \varepsilon \to 0,$$

$K_i \in \mathbb{C},\ i = 1, \ldots, k+1,$ *independent of* z_0*;* $H^f(z_0) \in \mathbb{C}$.

Sketch of the proof...

Asymptotic expansion of the directed area of the ε-neighborhood of an orbit

Here,

$$K_1 = \frac{k+1}{k} \cdot \sqrt{\pi} \cdot \frac{\Gamma(1+\frac{1}{2k+2})}{\Gamma(\frac{3}{2}+\frac{1}{2k+2})} \left(\frac{2}{|a_1|}\right)^{1/(k+1)} \cdot \nu_A,$$

$$K_{k+1} = \nu_A \cdot \left[\frac{1}{2(k+1)} Im(\lambda) + \left(\frac{k-1}{\pi(k+1)}\left(\frac{|a_1|}{2}\right)^{\frac{1}{k+1}} \frac{\frac{\Gamma(\frac{1}{2}+\frac{1}{2k+2})}{\Gamma(2+\frac{1}{2k+2})} - \sqrt{\pi}}{\frac{\Gamma(\frac{1}{k+1})}{\Gamma(\frac{3}{2}+\frac{1}{k+1})} + \sqrt{\pi}}\right) \cdot i \cdot Re(\lambda)\right].$$

Theorem (R, 2013)

Formal type (k,λ) explicitely from $(k; K_1, K_k)$ in finite expansion of ANY orbit!

Formal and analytic conjugacy

1. f and g **formally conjugated**

$$\exists \widehat{\varphi} \in z + z^2\mathbb{C}[[z]],\ g = \widehat{\varphi}^{-1} \circ f \circ \widehat{\varphi}$$

2. f and g analytically conjugated

$$\exists \varphi \in z + z^2\mathbb{C}\{z\},\ g = \varphi^{-1} \circ f \circ \varphi.$$

Toward analytic classification

Proposition (R)

The mapping

$$f \longmapsto \big(\varepsilon \mapsto A^{\mathbb{C}}(\varepsilon, z_0),\ \varepsilon \in (0, \varepsilon_0)\big)$$

is injective on the set of germs with z_0 in their attracting basin.

BUT

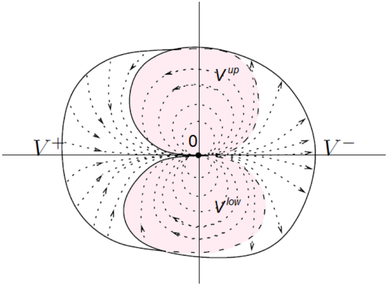
Proposition (R)

No asymptotic expansion of $R(z_0, \varepsilon)$, as $\varepsilon \to 0$, in power-log scale!

The reason. The critical index n_ε a jump function.

The simplest formal class

- $\star$ model formal class $(k = 1, \lambda = 0)$; $f_0 = Exp(z^2 \frac{d}{dz}) = \frac{z}{1-z}$
- $\star$ prenormalized $(a_1 = 1)$
- $\star$ $f(z) = z + z^2 + z^3 + o(z^3)$



Fatou coordinates and moduli of analytic classification

$\star$ *Ecalle, Voronin*: a **sectorially analytic** vector field s.t. f embeds on sectors in its flow, as time-one map (in general, not global)

$$\updownarrow$$

$\star$ Equation of the trivialisation of the flow (**Abel equation**):

$$\Psi(f(z)) - \Psi(z) = 1.$$

- unique (to a constant) formal solution $\widehat{\Psi}(z) \in -1/z + z\mathbb{C}[[z]]$,
- analytic solutions $\Psi_{\pm}(z)$ on $V_{\pm}$; asymptotic expansion $\widehat{\Psi}(z)$

$\rightarrow$ **Fatou coordinates, sectorial trivialisations**

Ecalle-Voronin moduli of analytic classification

On $V^{up},\ V^{low}$:

$$\Psi_+(f(z)) - \Psi_-(f(z)) = \Psi_+(z) - \Psi_-(z)$$

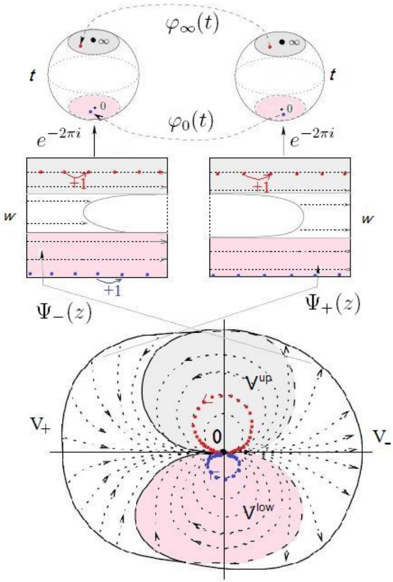
$\Rightarrow \Psi_+ - \Psi_-$ well-defined on *space of (closed) orbits* of $V^{up},\ V^{low}$

$\to$ lifts to poles of spaces of orbits of V_+
(spheres, $t = e^{2\pi i \Psi_+}$, orbits $\leftrightarrow$ points):

$$\begin{cases} \Psi_+(z) - \Psi_-(z) & = g_\infty(e^{2\pi i \Psi_+(z)}),\ z \in V^{up}, \\ \Psi_-(z) - \Psi_+(z) & = g_0(e^{-2\pi i \Psi_+(z)}),\ z \in V^{low}. \end{cases}$$

$\to$ a pair of analytic germs extended to 0,
$t \to (g_\infty(t), g_0(t))$,

$\to$ property $g_0(0) + g_\infty(0) = 0$.



$\varphi_\infty(t)$
∞
t
0
$\varphi_0(t)$
$e^{-2\pi i}$
$+1$
w
$\Psi_-(z)$
$\Psi_+(z)$
V^{up}
V_+
V_-
0
V^{low}

Ecalle-Voronin moduli of analytic classification

Ecalle-Voronin modulus of f: (g_∞, g_0), up to identifications:

$$(\star)\begin{cases}(g_1(t), g_2(t)) & \equiv (g_3(t), g_4(t)) \Leftrightarrow \\ & g_3(t) = g_1(t) + a,\ g_4(t) = g_2(t) - a, \\ & g_3(t) = g_1(bt),\ g_4(t) = g_2(t/b),\ a \in \mathbb{C},\ b \in \mathbb{C}^*.\end{cases}$$

Theorem (Ecalle-Voronin)

analytic classes of germs of the model formal type

$\updownarrow$

all pairs of analytic germs at $t = 0$,

$$(g_1(t), g_2(t)),\ g_1(0) + g_2(0) = 0,$$

up to identifications $(\star)$.

$\star$ analytic class of f_0 *trivial*: $(0, 0)$

Definition (R)

- $z \mapsto H^f(z),\ z \in V_+,$
  the **principal initial point dependent part for** f,
- $z \mapsto H^{f^{-1}}(z),\ z \in V_-,$
  the **principal initial point dependent part for** f^{-1}.

$$A^{\mathbb{C}}(\varepsilon, z) = A^{\mathbb{C}}(\varepsilon, f(z)) + z \cdot \varepsilon^2 \pi,\ \varepsilon \text{ small},$$

$$\overset{expansion}{\Longrightarrow} H^f(z) = H^f(f(z)) + z\pi$$

$*$ a *cohomological equation* similar to the Abel equation for f

$*$ *Stokes phenomenon*: sectorially analytic solutions?

Cohomological equations

- A **cohomological equation** for f:

$$H(f(z)) - H(z) = g(z),\ g(z) \in \mathbb{C}\{z\},\ g \not\equiv 0.$$

Sectorial solutions of cohomological equations *(Fatou, Loray)*

$g(z) = \alpha_0 + \alpha_1 z + O(z^2)$

- a unique *formal* solution $\widehat{H}(z) \in -\frac{\alpha_0}{z} + \alpha_1 Log(z) + z\mathbb{C}[[z]]$ (without the constant term),
- unique *sectorially analytic* solutions $H_\pm(z)$ on $V_\pm$, with expansion $\widehat{H}(z)$, $z \to 0$

Proof constructive!!!

Sectorial analyticity of principal parts

1-Abel equation for f: $H(f(z)) - H(z) = -z$
$\rightarrow$ the sectorial solutions $H_+,\ H_-$

Theorem (R)

- *the principal parts* $H^f(z)$ *i* $H^{f^{-1}}(z)$ *analytic on* $V_\pm$
- *explicitely related to solutions* $H_\pm(z)$ *of* 1*-Abel equation:*

$$\pi H_+(z) - \frac{\pi}{4} + i\pi^2 = H^f(z), \quad z \in V_+,$$
$$\pi H_-(z) - \frac{\pi}{4} = z - H^{f^{-1}}(z), \quad z \in V_-.$$

'*Global*' principal parts

Existence of global analytic solution H of cohomological equation
$\leftrightarrow H_+ - H_- \equiv 0\ (2\pi i)$ on $V^{up,low}$

1. global analytic solution of Abel equation

$$\Leftrightarrow f = \varphi^{-1} \circ f_0 \circ \varphi, \quad \varphi \in z + z^2\mathbb{C}\{z\}.$$

2. **Theorem (R)**

The 1*-Abel equation has a global analytic solution* $H(z)$
$\Leftrightarrow \ f(z) = \varphi^{-1}(e^z \cdot \varphi(z)), \ \varphi(z) \in z + z^2\mathbb{C}\{z\}.$

Germs with global solution to Abel and to 1-Abel equation

- $\mathcal{S} = \{f \mid f = \varphi^{-1}(e^z \cdot \varphi(z)),\ \varphi \in z + z^2\mathbb{C}\{z\}\}$
- $\mathcal{C}_0 = \{f \mid f = \varphi^{-1} \circ f_0 \circ \varphi,\ \varphi \in z + z^2\mathbb{C}\{z\}\}$

Example

1. $f_0(z) = \frac{z}{1-z} \in \mathcal{C}_0 \setminus \mathcal{S}$,

$$H_+(z) - H_-(z) = 2\pi i f_0(e^{-2\pi i \frac{1}{z}}),\ z \in V^{up},$$

$$H_-(z) - H_+(z) = -2\pi i + 2\pi i f_0(e^{2\pi i \cdot \frac{1}{z}}),\ z \in V^{low},$$

(explicitely computed by *Borel-Laplace transform*)

2. $f(z) = ze^z \in \mathcal{S} \setminus \mathcal{C}_0$,
3. $f(z) = -Log(2 - e^z) \in \mathcal{S} \cap \mathcal{C}_0$.

The sets $\mathcal{S}$ and $\mathcal{C}_0$ in general position $\Rightarrow$ the differences of sectorial solutions on petal intersections insufficient for determining the analytic class

Classifications of germs with respect to 1-Abel equation

$$H(f(z)) - H(z) = -z$$

$\Rightarrow (H_+ - H_-)(z) = (H_+ - H_-)(f(z))$, $z \in V^{up} \cup V^{low}$

$\Rightarrow H_+ - H_-$ *constant along orbits*

$$H_+ - H_- = g_\infty(e^{2\pi i \Psi_+(z)}),\ \ z \in V^{up},$$
$$H_- - H_+ = -2\pi i + g_0(e^{-2\pi i \Psi_+(z)}),\ \ z \in V^{low}.$$

$\Rightarrow (g_\infty(t), g_0(t))$, $g_\infty(0) + g_0(0) = 0$ a pair of analytic germs

Definition (R)

- The 1-**moment of** f: the pair (g_∞, g_0), up to identifications
- 1-**conjugacy class of** f: $[f]_1$

1-conjugacy classes vs. analytic classes

Theorem (Realization of 1-moments. Transversality, (R))

(g_0, g_∞) *a pair of analytic germs s.t.* $g_0(0) + g_\infty(0) = 0$. *Then:*

- *There exists a germ in the model formal class such that the given pair is its* 1*-moment.*
- *Moreover, such germ exists inside ANY analytic class.*

The results published in papers:

- M. Resman, *ε-neighborhoods of orbits and formal classification of parabolic diffeomorphisms*, Discrete Contin. Dyn. Syst. **33**, 8 (2013), 3767–3790
- M. Resman, *ε-neighborhoods of orbits of parabolic diffeomorphisms and cohomological equations*. Nonlinearity 27 (2014), 3005–3029

Thank you for the attention!