Control and optimization techniques for "jerk" type circuits

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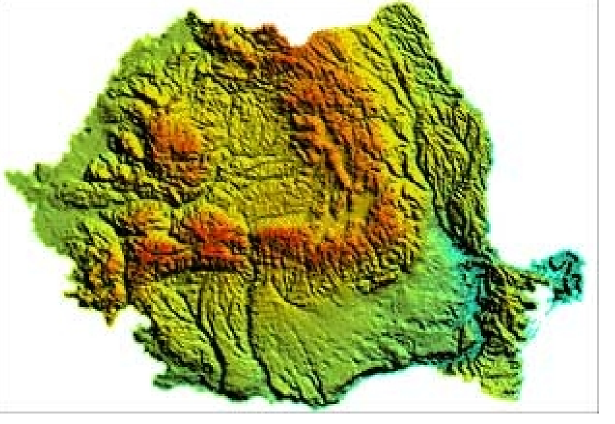

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I. SYMMETRY AND INTEGRABILITY. KEY ASPECTS

I.1. Integrability of the dynamical systems

- Dynamical systems are usually described through nonlinear differential equations. If solutions exist, the diffential system is said to be *integrable*.
- In the case of autonomous Hamiltonian systems: *integrability* means the existence of some analytical and time-independent quantities $\{C_i, i = 1,\dots n\}$ in involution (*n* the number of degrees of freedom):

$$\frac{dC_i}{dt} = \{H, C_i\} = 0; \{C_i, C_j\} = 0 \qquad (1)$$

- There is no a general theory/procedure allowing to completely solve nonlinear PDEs. Sometimes it is quite enough to decide if the system is integrable or not. *Methods*:

  1) Hirota' s bilinear method; 2) Backlund transformation;
  3) Inverse scattering; 4) Lax pair operators;
  5) Painleve analysis; 6) *Symmetry approach*, etc.

- *The symmetry method* - efficient techniques in studying integrability. It allows to obtain:
  - The first integrals/invariants specific for symmetry transformations;
  - Classes of exact solutions through *similarity reduction* (reduction of PDEs to ODEs).
  - New solutions starting from known ones.

I.2. Point-like symmetries. Lie operators.

- Let us consider a system of q partial differential equations (PDEs):

$$\Delta = \{\Delta^{\nu}(t, x, u(x,t), u^{(n)}(x,t))\}_{\nu=1}^{q} = 0 \quad (1.1)$$

defined on a domain $M \subset R^{p}$ (i.e. a connected open subset of R^{p}) with at most $n-$ th order partial derivatives of $u(x,t) = \{u^{1}(x,t), \dots, u^{q}(x,t)\}$ in the space-time variables $(x,t) = \{t, x^{1}, \dots, x^{p}\}$. The notation $u^{\alpha(J)}(x,t)$ designates the partial derivatives of $\{u^{\alpha}(x,t), \alpha = 1, \dots q\}$ up to the J - th order:

$$u^{\alpha(J)} = \frac{\partial^{J} u^{\alpha}}{\partial t^{j_0} \partial x^{1(j_1)} \partial x^{2(j_2)} \dots \partial x^{p(j_p)}} \equiv D^{J} u^{\alpha};\ J = j_0 + j_1 + \dots + j_p \quad (1.2)$$

- A *point-like transformation* may be defined through an infinitesimal parameter *ε* by:

$$\begin{aligned} & t' = t + \delta t,\ \delta t = \varepsilon \varphi(x,t) + O(\varepsilon^{2}) + \dots \\ & x = \{x^{i}, i = 1, \dots, p\}; x' = \{x'^{i}, i = 1, \dots, p\} \\ & x'^{i} = x^{i} + \delta x^{i},\ \delta x^{i} = \varepsilon \cdot \xi^{i}(x,t) + O(\varepsilon^{2}) + \dots \end{aligned} \quad (1.3)$$

- The transformations (1.3) induce a first order variation of the dependent variables given by:

$$\delta u = u'(x', t') - u(x,t) = \frac{\partial u}{\partial t} \delta t + \sum_{i=1}^{p} \frac{\partial u}{\partial x_i} \delta x_i \equiv \varepsilon \cdot U \cdot u(x,t) \quad (1.4)$$

- The operator *U* denotes the generator of the infinitesimal point-like transformations and is called *Lie operator*. In the first order approximation its concrete form is:

$$U=\varphi\frac{\partial}{\partial t}+\sum_{i=1}^{p}\xi^{i}(t,x,u)\frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{q}\phi^{\alpha}(t,x,u)\frac{\partial}{\partial u^{\alpha}} \qquad (1.5)$$

- Let us denote by $U^{(n)}$ the n-th order extension of the Lie infinitesimal symmetry operator:

$$U^{(n)}=U+\sum_{\alpha=1}^{q}\sum_{J}\phi^{\alpha(J)}(x,u^{(n)})\frac{\partial}{\partial u^{\alpha(J)}}$$
$$\phi^{\alpha(J)}(x,u^{(n)})=D^{J}[\phi^{\alpha}-\sum_{i=1}^{p}\xi^{i}u_{i}^{\alpha}]+\sum_{i=1}^{p}\xi^{i}u_{i}^{\alpha(J)},\quad u_{i}^{\alpha}=\frac{\partial u^{\alpha}}{\partial x^{i}},\alpha=1,\dots q \qquad (1.6)$$

- Lie symmetry method requires to impose the following invariance condition:
  [P.J.Olver- Applications of Lie Groups to Differential Equations, GTM 107, Second edition, Springer-Verlag, 1993]

$$\Delta'\equiv U^{(n)}(\Delta)\big|_{\Delta=0}=0\ \ for\,\Delta\equiv\{\Delta^{\nu},\ \nu=1,\dots q\} \qquad (1.7)$$

Within (1.7) the equations (1.1) could change their form but not the class of solutions.

- **CONCLUSIONS:**
  - For each PDE there is a local group of transformations on the space of its independent and dependent variables called symmetry group that maps the set of all analytical solutions on itself.
  - Knowledge of Lie symmetries allows the construction of the group-invariant solutions.

I.3. Invariants and similarity reduction

- One of the advantages of the method: find solutions of the original PDEs by solving ODEs. These ODEs, called *reduced equations*, are obtained by introducing suitable new variables, determined as invariant functions in respect to the Lie generators.

- By applying Lie operators on the equations, one get the *determining system*. It allows to effectively find the symmetry generators $\{\varphi(t,x,u), \xi^i(t,x,u), \phi^\alpha(t,x,u)\}$

- Knowing the symmetry generators we have to solve the associated *characteristic equations*:

$$\frac{dt}{\varphi} = \frac{dx^1}{\xi^1} = \ldots = \frac{dx^p}{\xi^p} = \frac{du^1}{\phi_1} = \ldots = \frac{du^q}{\phi_q} \qquad (1.8)$$

- By integrating, we obtain the invariants of the analyzed system $\{I_r, r = 1,\ldots, p+q\}$.

- *Similarity reduction*: the invariants are chosen as *similarity variables* and they are expressed in terms of the original ones: $p+1$ independent variables and q dependent. We get a set of differential equations with only $(p+q)$ variables.

I.3.1 Generalization *of the Lie symmetry method*

1. The *non-classical symmetry method* (Bluman and Cole): added the invariance surface condition:

$$Q^{\alpha}(x,u^{(1)}) \equiv \phi_{\alpha}(x,u) - \sum_{i=1}^{p} \xi^{i}(x,u)\frac{\partial u^{\alpha}}{\partial x^{i}} = 0,\ \alpha = \overline{1,q} \qquad (1.9)$$

   Consequences:

   - Smaller number of determining equations for the infinitesimals $\xi^{i}(x,u),\ \phi_{\alpha}(x,u)$.
   - More solutions than the CSM (any classical symmetry is a non-classical one)

2. The *direct method* (Clarkson and Kruskal): a direct, algorithmic method for finding symmetry reductions.

3. The *differential constraint approach* (Olver and Rosenau): the original system of partial differential equations can be enlarged by appending additional differential constraints (side conditions), resulting an over-determined system of partial differential equations.

4. The *generalized conditional symmetries method* or *conditional Lie-Bäcklund symmetries*
   [Fokas, Liu- Theor. Math. Phys.99 , 571 (1994) and Zhdanov- J. Phys. A: Math. Gen. 28, 3841(1995)].

I.4.The inverse symmetry problem

- *The* ***direct symmetry problem*** for evolutionary equations consists in:
  - Determining the Lie symmetry group corresponding to a given evolutionary equation.
  - Obtaining the invariants associated to each symmetry operator.
  - Obtaining some reduced equations with the similarity reduction procedure.
  - Solving the reduced equation and generating similarity solutions of the model.
- The ***inverse symmetry problem***: what is the largest class of evolutionary equations which are equivalent from the point of view of their symmetries?
- **Example** of a $2D$ dynamical system:

$$u_t = A(x,y,t,u)u_{xy} + B(x,y,t,u)u_x u_y + C(x,y,t,u)u_{2x} + D(x,y,t,u)u_{2y} + \\ + E(x,y,t,u)u_y + F(x,y,t,u)u_x + G(x,y,t,u) \quad (1.10)$$

- The general expression of the Lie symmetry operator with $\varphi \equiv 1$:

$$U(x,y,t,u) = \varphi(x,y,t,u)\frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u} \quad (1.11)$$

- The symmetry invariance condition is given by the relation:

$$0 = U^{(2)}[u_t - A(x,y,t,u)u_{xy} - B(x,y,t,u)u_x u_y - C(x,y,t,u)u_{2x} - D(x,y,t,u)u_{2y} - \\ - E(x,y,t,u)u_y - F(x,y,t,u)u_x - G(x,y,t,u)]$$

- The previous relation has the equivalent expression:

$$0 = -A_t u_{xy} - B_t u_x u_y - C_t u_{2x} - D_t u_{2y} - E_t u_y - F_t u_x - G_t - A_x \xi u_{xy} - B_x \xi u_x u_y -$$
$$-C_x \xi u_{2x} - D_x \xi u_{2y} - E_x \xi u_y - F_x \xi u_x - G_x \xi - A_y \eta u_{xy} - B_y \eta u_x u_y - C_y \eta u_{2x} - D_y \eta u_{2y} -$$
$$-E_y \eta u_y - F_y \eta u_x - G_y \eta - A_u \phi u_{xy} - B_u \phi u_x u_y - C_u \phi u_{2x} - D_u \phi u_{2y} - E_u \phi u_y - F_u \phi u_x -$$
$$-G_u \phi + \phi^t - A\phi^{xy} - C\phi^{2x} - D\phi^{2y} - B\phi^x u_y - F\phi^x - B\phi^y u_x - E\phi^y$$

- Equating with zero the coefficients of various monomials in derivatives of u, we get 11 equations:

$$0 = \xi_u;\ 0 = \eta_u;\ 0 = B\eta_x - D\phi_{2u};\ 0 = B\xi_y - C\phi_{2u}$$
$$0 = A\eta_y - \eta A_y - A_u\phi + A\xi_x - \xi A_x + +2D\xi_y + 2C\eta_x - A_t$$
$$0 = A\eta_x + 2D\eta_y - \eta D_y - \xi D_x - D_u\phi - D_t$$
$$0 = -A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_t - B_x\xi - B_u\phi - B_y\eta$$
$$0 = -\eta_t + F\eta_x - B\phi_x + E\eta_y - E_t - E_x\xi - E_y\eta - E_u\phi + A\eta_{xy} - A\phi_{xu} + C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu}$$
$$0 = -\xi_t - B\phi_y + F\xi_x + E\xi_y - F_t - F_x\xi - F_y\eta - F_u\phi$$
$$A\xi_{xy} - A\phi_{yu} + C\xi_{2x} + D\xi_{2y} - 2C\phi_{xu}$$
$$0 = \phi_t + G\phi_u - F\phi_x - E\phi_y - G_t - G_x\xi - G_y\eta - G_u\phi$$
$$-A\phi_{xy} - C\phi_{2x} - D\phi_{2y}$$

Results: [R. Cimpoiasu, R. Constantinescu, Nonlin. Analysis:Theory, Methods and Applications, vol.73, Issue1, 2010, 147]

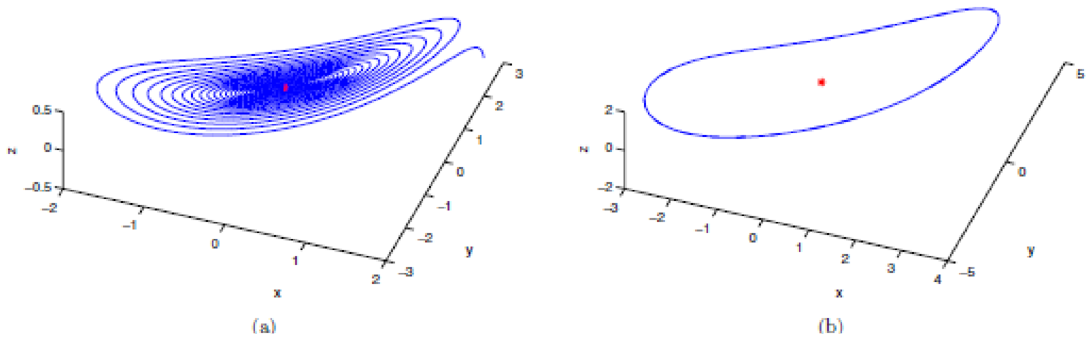
1. In the linear sector for $\varphi(t,x,u), \xi(x,y,t,u), \eta(x,y,t,u), \phi(x,y,t,u)$, the maximal degree of the Lie Algebra is 8. It is not nilpotent but it is a solvable algebra.

   2. The nonlinear heat equation and the transfer equation with power-law nonlinearities belong to the same class as their symmetries are concerned: $u_t = (g(u)u_x)_y \Leftrightarrow u_t = \partial_x(\alpha x^s u_x) + \partial_y(\beta y^r u_y) + f(u)$

II. CHAOS AND ITS CONTROL. APPLICATIONS

II.1. What control and optimization mean?

- **The control theory** is a very developed branch of nonlinear sciences, extremely important in many fields and with a lot of specific procedures proposed during the time. It supposes that, starting from a system with non-regular dynamics, we can "optimize" its evolution, or to "control" the original system.



Unified Lorenz-Type System [Q.Yang, Y.Chen Int.J.Bifurc&Chaos, Vol 24 (2014), 450055]

- The first article on chaos control was published in 1989 by Hubler. In 1990 Pecora and Carroll proposed the idea of chaos synchronization. In the last years, many techniques for chaos control and synchronization have been developed:

  1) feedback method — 2) adaptive technique
  3) time delay feedback approach — 4) active method, etc.

- We will start from a procedure proposed for Hamiltonian systems and we will try to extend it towards more pragmatic non-variational systems appearing in electronics.

II.2. Chaos control for Hamiltonian systems

- Let us consider an integrable system described by the Hamiltonian H_0 and a chaotic system described by the "perturbed" Hamiltonian of the form:

$$H' = H_0 + V_1 \tag{2.2.1}$$

The problem of controlling the chaos is the following: to find a **control term** V_2 such that the dynamics of the "controlled" Hamiltonian $H_0 + V_1 + V_2$ becomes more regular than of the perturbed system.

We will follow an algorithm proposed in [*Ciraolo, G., Chandre, C., Lima, R., Vittot, M., Pettini, M., Figarella, C. and Ghendrih, Ph.: 2003, "Control of chaos in Hamiltonian systems", archived in arXiv.org/nlin.CD/0311009*].
It allows finding the control term as a series whose items can be explicitly and easily computed by recursion.
Let $\mathbf{A}$ be the algebra of the real functions defined on the phase space. For $V \in \mathbf{A}$ the time evolution following the flow of the time independent H is given by the equation

$$\frac{dV}{dt} = \{H,V\} \equiv \{H\}V \tag{2.2.2}$$

We introduced the notation:

$$\{H\}* \equiv \{H,*\} \tag{2.2.3}$$

The eq. (2.2.1) is formally solved as

$$V(t) = e^{t\{H\}}V(0) \tag{2.2.4}$$

- The vector space $Ker\{H\}$ is the set of constants of motion, that is:

$$Ker\{H\} = \{C \in \mathbf{A}; \{H,C\} = 0\},\ Ker\{H\} \subset \mathbf{A} \tag{2.2.5}$$

- Three new operators are defined in connection with the operator $\{H_0\}$, attached to the initial integrable Hamiltonian H_0:

i) The *pseudo-inverse* operator Γ of $\{H_0\}$ such that $\{H_0\}^2\Gamma = \{H_0\}$ which is equivalent with:

$$\{H_0,\{H_0,\Gamma V\}\} = \{H_0,V\}; \tag{2.2.6}$$

ii) The *non-resonant* part $\mathbf{N}$ of $\{H_0\}$ with the action on the algebra $\mathbf{A}$ of the form:

$$\mathbf{N}V = \{H_0,\Gamma V\} \Leftrightarrow \{H_0,\mathbf{N}V\} = \{H_0,V\},(\forall) \quad V \in \mathbf{A}; \tag{2.2.7}$$

iii) The *resonant* part $\mathbf{R}$ of $\{H_0\}$ such that $\mathbf{R} = 1 - \mathbf{N}$ which is equivalent with:

$$\{H_0,\mathbf{R}V\} = 0, \quad (\forall) \quad V \in \mathbf{A}. \tag{2.2.8}$$

- A control term V_2 for the perturbed Hamiltonian $H_0 + V_1$ is determined such that $H_0 + V_1 + V_2$ is canonically conjugated with $H_0 + \mathbf{R}V_1$.

This control condition implies that the flow of $H_0 + V_1 + V_2$ is conjugated with the flow of $H_0 + \mathbf{R}V_1$, therefore the following relation is satisfied:

$$e^{t(H_0+V_1+V_2)} = e^{-\{\Gamma V_1\}} e^{t(H_0+\mathbf{R}V_1)} e^{\{\Gamma V_1\}} \tag{2.2.9}$$

The analytical expression of V_2 is:

$$V_2(V_1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \{\Gamma V_1\}^n (n\mathbf{R}+1)V_1 \tag{2.2.10}$$

Definition:

Ciraolo, G., Chandre, C., Lima, R., Vittot, M., Pettini, M., Figarella, C. and Ghendrih, Ph.: 2003, "Control of chaos in Hamiltonian systems", archived in arXiv.org/nlin.CD/0311009.

The Hamiltonian H_0 is non-resonant if and only if, for any action variable $A \in \mathbf{B} \subset \mathbf{R}^l$ and for the frequency vector $\omega(A) = \frac{\partial H_0}{\partial A}$, the relation $\omega(A) \cdot k = 0$ implies $k = 0$.

Remarks:

- If H_0 is non-resonant, then the Hamiltonian $H_0 + \mathbf{R}V_1$ is integrable. If H_0 is resonant and $\mathbf{R}V_1 = 0$, then the controlled Hamiltonian H is conjugated with the integrable H_0.

- In the case $\mathbf{R}V_1 = 0$, the expansion (2.2.10) of the control term can be written as:

$$V_2(V_1) = \sum_{s=2}^{\infty} (V_2)_s, \; (V_2)_s = -\frac{1}{s}\{\Gamma V_1, (V_2)_{s-1}\}, \; (V_2)_1 = V_1 \qquad (2.2.11)$$

- In the previously presented theory, if we introduce a parameter ε so that $V_1 \approx \varepsilon$, usually V_2 is a second order quantity $(\approx \varepsilon^2)$. Bellow we will see that control terms V_2 of the same order as the perturbation V_1 are also available. Such terms can be found by the invariant ''deformation''.

II.3. Yang-Mills mechanical model

Yang-Mills theory of the nonabelian gauge field in $3+1$ Minkowski space is described by:

$$S_0[A^m_\mu] = -\frac{1}{4}\int d^4x\, F^m_{\mu\nu} F^{\mu\nu}_m \quad \text{where} \quad F^m_{\mu\nu} = \partial_\mu A^m_\nu - \partial_\nu A^m_\mu + g\varepsilon^m_{nr} A^n_\mu A^r_\nu .$$

The Euler-Lagrange equation:

$$\partial_\mu F^{\mu\nu}_m + g\varepsilon_{mnr} A^n_\mu F^{\mu\nu r} = 0 \qquad (2.3.1)$$

The fields can be expressed in terms of a finite set of colour factors imposing that:

$$A^0_m = 0,\ \partial^j A^i_m = 0,\ A^i_m(t) = \frac{1}{g} O^i_m f^{(m)}(t),\ O^i_m O^i_n = \delta_{mn} \qquad (2.\ 3.2)$$

One obtains the system of "mechanical" equations

[S. G. Matincan, G. K. Savvidi, N. G. Ter-Arutyunyan-Savvidi, Sov. Phys. JETP 53 (3) (1981) 421]:

$$\ddot{f}^{(m)} + f^{(m)}(\mathbf{f}^2 - f^{(m)2}) = 0; m = 1,2,3 \qquad (2.\ 3.3)$$

With the notations $f^{(1)} \equiv x, f^{(2)} = f^{(3)} = y$ and choosing some arbitrary coefficients we have:

$$\ddot{x} = -x(3 + 2x^2 + \frac{1}{2}y^3)$$
$$\ddot{y} = -y(1 + \frac{1}{2}x^2 + 4y^2) \qquad (2.\ 3.4)$$

Integrability cases for the 2D Yang-Mills Model

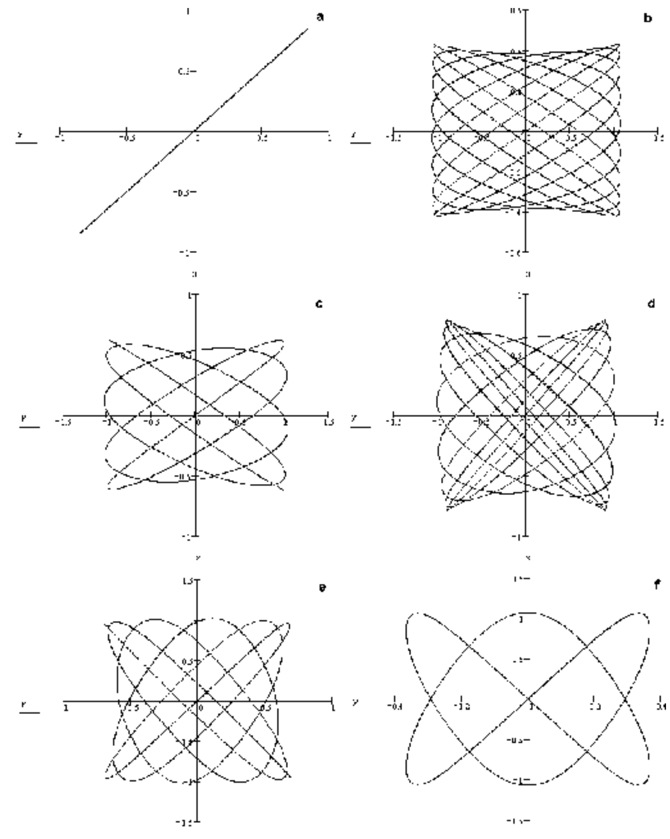
- Let us consider the *generalized Yang-Mills model* in 2 dimensions generated by the Hamiltonian:

$$H(x,y)=\frac{1}{2}(\dot{x}^2+\dot{y}^2)-\frac{A}{2}x^2-\frac{B}{2}y^2+ax^2y^2+bx^4+dy^4 \qquad (2.\ 3.5)$$

Where *A, B, a, b, d* are parameters. The equations of motion have the forms:

$$\ddot{x}=-\frac{\partial H}{\partial x}=Ax-2axy^2+4bx^3$$

$$\ddot{y}=-\frac{\partial H}{\partial y}=By+2ax^2y+4dy^3$$



- Despite it looks simple, the system is integrable in only 4 cases.

- It describes a chaotic behavior, but traces of regularity have been found.

[R.Cimpoiasu, R.Constantinescu, J.Nonlin.Math.Phys., vol **13**, no. 2, (2006), 285-292]

Chaos control for YM Systems

- Let $H_0(x, y, \dot{x}, \dot{y})$ be a Hamiltonian admitting a second invariant $C_0(x, y, \dot{x}, \dot{y})$. Let $V_1 = V_1(x, y)$ be a perturbation so that $H_0 + V_1$ is nonintegrable. We look for a control term $V_2(x, y)$ and for a deformation of C_0, denoted $P(x, y)$, so that $H_0 + V_1 + V_2$ be integrable with the second integral of the form:

$$C(x, y, \dot{x}, \dot{y}) \equiv C_0(x, y, \dot{x}, \dot{y}) + P(x, y) \qquad (2.3.6)$$

The invariance condition $\{H, C\} = 0$ imposes:

$$\dot{x}\frac{\partial P}{\partial x} + \dot{y}\frac{\partial P}{\partial y} - \frac{\partial V_1}{\partial x}\frac{\partial C_0}{\partial \dot{x}} - \frac{\partial V_1}{\partial y}\frac{\partial C_0}{\partial \dot{y}} - \frac{\partial V_2}{\partial x}\frac{\partial C_0}{\partial \dot{x}} - \frac{\partial V_2}{\partial y}\frac{\partial C_0}{\partial \dot{y}} = 0 \qquad (2.3.7)$$

We know V_1 and we determine V_2 and P.

- Let us consider the integrable system described by the Hamiltonian:

$$H_0 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}x^2 + \frac{1}{8}y^2 \qquad (2.3.8)$$

The second invariant is:

$$C_0 = x\dot{y}^2 - y\dot{x}\,\dot{y} - \frac{1}{4}xy^2 \qquad (2.3.9)$$

We choose the following perturbation:

$$V_1 = xy^2 \tag{2.3.10}$$

and we look for a control term of the form:

$$V_2 = a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 \tag{2.3.11}$$

With this expression in (2.3.7), we find that we should have: $a_2 = 0; a_3 = 0; a_4 = \frac{1}{2}a_1 - 1$

and:

$$P = -\frac{1}{8}a_1y^4 - \frac{1}{2}a_1x^2y^2 \tag{2.3.11}$$

In *Fig. 1* and *Fig. 2* the Poincare sections respectively for the chaotic and controlled system for which the "perturbed" system is described by the Hamiltonian (2.3.8) are presented. It is obvious that the dynamic of the controlled system is more regular than that of the original one.

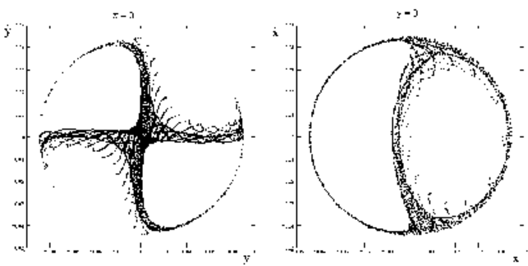


Fig.1: Perturbed system

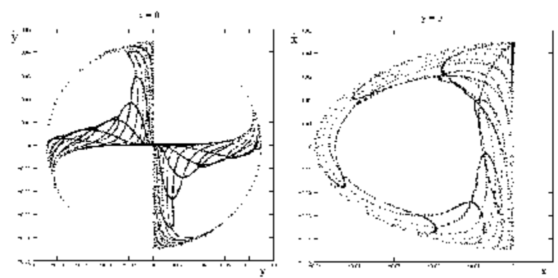


Fig.2: Controlled system

II.4. “Jerk” equations

- “Jerk equations” represent third order differential equation. The term “jerk” was proposed by Schot (1978) as a name for the quantity describing the variation of the acceleration in a mechanical system.

- Why Jerk eqs. are important in the chaos context?
  They are the lowest order ODEs with smooth continuous functions which can give chaos.

- What is the simplest Jerk which generate chaos?
  First study: Hans Gottlieb (1996) who considered (only) eqs. of the form $\dddot{x} = f(x, \dot{x}, \ddot{x})$
  Rational systems were also found.

- Which systems with physical significance can be associated with Jerk?
  Any Jerk can be cast in the form of a system of three coupled first-order ODEs, but not conversely.

Examples of quadratic systems: [Simin Yu, Jinhu Lu, Wallace Tang, Guanrong Chen–Chaos, Vol 16, 033126(2006)]

$$\frac{dx}{d\tau} = a_1 x + a_2 y + a_{13} xz + a_{23} yz,$$

$$\frac{dy}{d\tau} = b_1 x + b_2 y + b_{13} xz + b_{23} yz + d_2,$$

$$\frac{dz}{d\tau} = c_3 z + c_{12} xy + c_{11} x^2 + c_{22} y^2 + c_{33} z^2 + d_3,$$

a_1	a_2	a_{13}	a_{23}	b_1	b_2	b_{13}	b_{23}	d_2	c_3	c_{12}	c_{11}	c_{22}	c_{33}	d_3	System
−10	10	0	0	28	−1	−1	0	0	$-\frac{8}{3}$	1	0	0	0	0	Lorenz
−35	35	0	0	−7	28	−1	0	0	−3	1	0	0	0	0	Chen
−36	36	0	0	0	20	−1	0	0	−3	1	0	0	0	0	Lü
2.86	0	0	−1	0	−10	1	0	1	−4	1	0	0	0	0	Lorenz-like
0.5	0	0	1	0	−10	−1	0	0	−4	−1	0	0	0	0	Liu-Chen
−2	6.7	0	−1	1	0	0	0	0	−1	0	0	1	0	0	Ruchlidge
0	1	0	0	1	−0.85	−1	0	0	−0.5	0	1	0	0	0	S-M
0	1	0	0	−1	0	0	1	0	0	0	0	−1	0	1	Sprott (I)
0	0	0	1	1	−1	0	0	0	0	−1	0	0	0	1	Sprott (II)
0	0	0	1	1	−1	0	0	0	0	0	−1	0	0	1	Sprott (III)

II.5. Application: Chua circuit

II.5.1. Chua system of equations

- **Chua circuit** is a specific type of electronic circuit containing a diode with a nonlinear intensity-voltage characteristic.
- The circuit becomes a chaotic oscillator, generating stochastic signals, with important applications in communication technologies, biology, neurosciences, and in other fields.

$$\frac{dV_1}{dt} = \frac{1}{RC_1}(V_2 - V_1) - \frac{1}{C_1}g(V_X)$$

$$\frac{dV_2}{dt} = \frac{1}{RC_2}(V_1 - V_2) + \frac{1}{C_2}I_L$$

$$\frac{dI_L}{dt} = -V_L \equiv V_2; V_X \equiv V_1$$

The characteristic $I_X = g(V_X)$ of Chua diode is strongly nonlinear, but chosen to be symmetric in respect with positive or negative values of the potential. Leon Chua approximated the diode's conductance by a linear function with three anti-symmetric segments:

$$g(V_1) = AV_1 + \frac{1}{2}(A-B)[|V_1+1| - |V_1-1|]$$

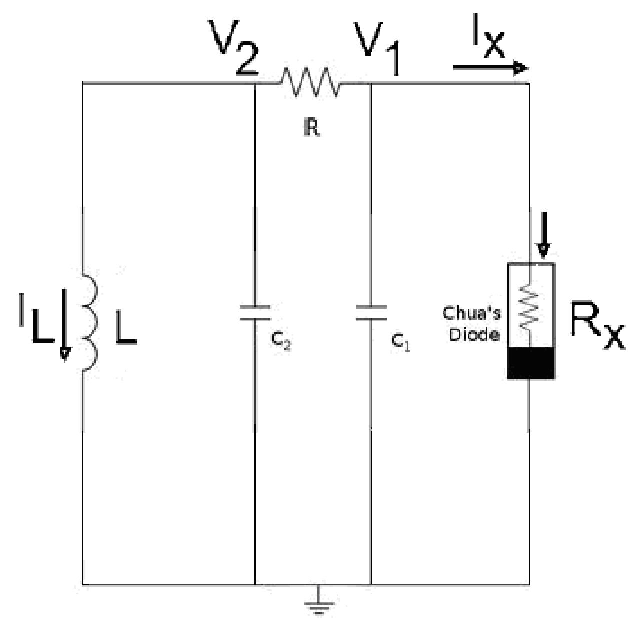


Fig. 1 Chua's electronic circuit

II.5.2. Chaos and symmetry for Chua system

By convenient notations we get:

$$\dot{x} = \alpha(y - x - f(x))$$
$$\dot{y} = x - y + z$$
$$\dot{z} = -\beta y.$$

- **NOTE 1:** The system is similar with Lorenz:

$$\dot{x} = a(x - y)$$
$$\dot{y} = bx + cy - xz$$
$$\dot{z} = mz + xy$$

- **NOTE 2:** Chua system admits a "jerk" representation of the form:

$$\dddot{x} + [\dot{f}(x) + 1]\ddot{x} + \ddot{f}(x)\dot{x}^2 + [\dot{f}(x) + \beta - \alpha]\dot{x} + \beta f(x) = 0.$$

We firstly computed the Lie symmetries with:

$$X^{(3)}(t,x) = \varphi\partial_t + \phi\partial_x + \phi^t\partial_{\dot{x}} + \phi^{2t}\partial_{\ddot{x}} + \phi^{3t}\partial_{\dddot{x}}$$

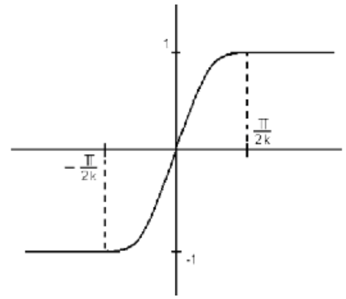
We got five cases for which Chua system admits invariant solutions:

1) $f(x) = c_1 x + c_2$ 2) $f(x) = \frac{c_4}{2}x^2 + (c_1c_4 - 1 + 6c_2)x + c_5$ 3) $f(x) = -\frac{3}{c_2}x^2 - (1 + \frac{6c_1}{c_2})x + c_3$

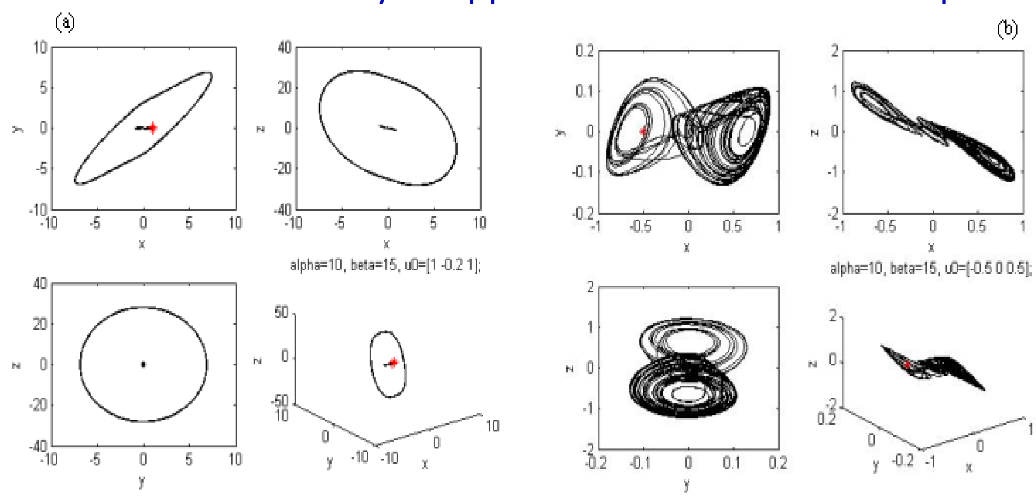
4) $f(x) = -x + \ln(c_1 + x)c_2 + \ln(c_1 + x)c_1 + c_3$ 5) $f(x) = \frac{e^{c_4c_5}(x + c_1)^{(1-c_4)}}{c_4(1 - c_4)} - \left(1 + \frac{3c_2}{c_4} - 3c_2\right)x + c_6$

- We investigated the chaotic behavior of the system for:

$$f(x) = \begin{cases} \sin kx, & x \in \left[-\frac{\pi}{2k}, \frac{\pi}{2k}\right] \\ -1, & x < -\frac{\pi}{2k} \\ 1, & x > \frac{\pi}{2k} \end{cases}$$



- Depending on the values of the parameters α and β, the system has a dual dynamical behavior: chaotic and regular orbits.
- No Hopf bifurcation and no limit cycle appear because there are not pure imaginary roots.



(a) Limit cycle for Chua system in case alpha = 10, beta =15, with initial conditions u0 = [1, -0.2, 1] and (b) chaos for the case alpha = 10, beta =15 and initial conditions u0 = [-0.5, 0, 0.5]

II.5.3. A “Jerk” version of Chua system

Let us consider a jerk equation of the form:

$$\dddot{x} + \beta\,\ddot{x} + \gamma\dot{x} = f(x)$$

It can be re-written as:

$$\dot{x} = y \Rightarrow \ddot{y} + \beta\,\dot{y} + \gamma\, y = f(x)$$

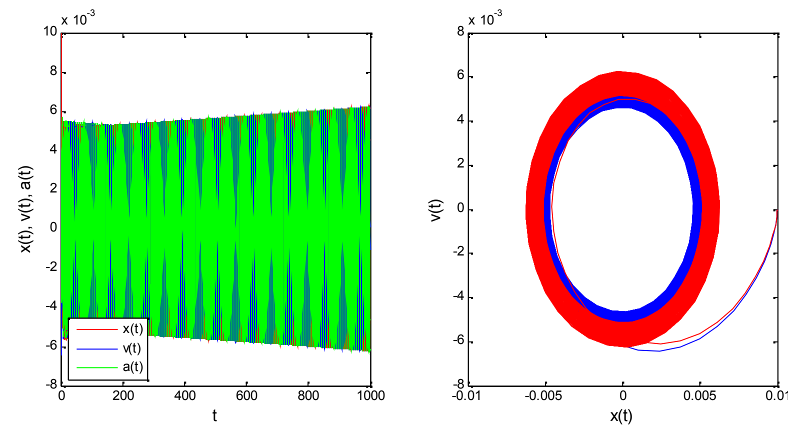
$$\dot{y} = z \Rightarrow \dot{z} + \beta\, z + \gamma\, y = f(x)$$

We will investigate the case: $f(x) = thx$

For various parameters α, β the system is chaotic. We control the system with a quadratic term:

$$f(x) = A[th(x + A) + th(x - A)] + Bx^2$$

The system starts to behave as a regular one. It has an attractor, so it is asymptotic stable.



$greenx_0 - 0.01; y_0 = 0; z_0 = 0; \gamma = 1, \beta = 0.6; A = 1.748; B = 0(red); B = 5(blu)$

II.5.4. Chua as a variational (Lagrangean) system

Can we find a Lagrangian for a non-variational system?

For second order ODE there is a nice technique based on the Jacobi Last Multiplier.

[Nucci MC and Leach PGL *Some Lagrangians for Systems without a Lagrangian*, Phys. Scripta 83 (2011) 035007]

We extended this technique for Chua third order equation (Jerk):

$$\dddot{x}+\alpha\ddot{x}+\alpha\frac{d^2 f(x)}{dt^2}+\ddot{x}+\alpha\frac{df(x)}{dt}=-\beta(\dot{x}+\alpha x+\alpha f(x))$$

By using the notation:

$$F(t,x,\dot{x})=-\frac{\dot{x}^2}{x}-\left(\alpha+1+\alpha\frac{df(t)}{dt}\right)\frac{\dot{x}}{x}-\left(\frac{1}{x}\beta+\alpha\frac{1}{x}\frac{df(t)}{dt}+\alpha\frac{d^2 f(t)}{dt^2}\right)-\alpha\beta\frac{t}{x^2}-\alpha\beta\frac{f(t)}{x^2}$$

We get for the last multiplier the equation:

$$\frac{d}{dt}(\ln M)=-\frac{\partial F}{\partial\dot{x}}$$

The Lagrangian attached to the system can be written in terms of two arbitrary functions as:

$$L=\int d\dot{x}\int d\dot{x}M+\frac{dG(t,x)}{dt}+f_3(t,x)$$

Result: Chua system admits a first integral of the form:

$$\mathtt{I}=\left(\frac{\alpha\beta(\alpha-1)}{\alpha+\beta}+\ln y\right)x-\left(\frac{\alpha\beta(\beta+1)}{\alpha+\beta}-\ln y\right)z-y$$

SELECTED REFERENCES

- P. J.Olver, ,,Applications of Lie Groups to Differential Equations, GTM 107, Second edn., Springer-Verlag, 1993.
- Bluman G W and Kumei S, Symmetries and Differential Equations (New York: Springer), 1989.
- Nucci M.C. and Clarkson P.A., Phys. Lett. A **184**,1992 ,49-56.D.J. Arrigo, P. Brosdbridge and J.M. Hill, Nonclassical symmetry solutions and the methods of Bluman-Cole
- Arrigo D.J., Brosdbridge P. and Hill J.M., J. Math. Phys. **34** (l0), 1993, 4692-4703.
- Levi D. and Winternitz P., J. Phys. A: Math. Gen. **22**, 1989, 2915-2924.
- Pucci E., Similarity reductions of partial differential equations, J. Phys. A **25**, 2631-2640.1992.
- Clarkson P A and Kruskal M D, J. Math. Phys.**30**, 1989, 2201--13.
- Ovsiannikov L.V., Group Analysis of Differential Equations, Academic Press, New York (1982).
- Ruggieri M. and Valenti A., Proc. WASCOM 2005, R. Monaco, G. Mulone, S. Rionero and T. Ruggeri eds., World Sc. Pub., Singapore, (2006),481.
- **R. Cimpoiasu, R. Constantinescu, Nonlinear Analysis:Theory, Methods and Applications, vol.**73**, Issue1, 2010, 147-153.**
- I.Bakas, Renormalization group flows and continual Lie algebras, JHEP **0308**, 013-(2003), hep-th/0307154.
- A.F.Tenorio, Acta Math. Univ. Comenianae, Vol. **LXXVII**, 1(2008),141--145.
- A. Ahmad, Ashfaque H. Bokhari, A.H. Kara and F.D. Zaman, J. Math. Anal. Appl. **339**, 2008, 175-181**.**
- **R. Cimpoiasu., R. Constantinescu, Nonlinear Analysis Series A: Theory, Methods & Applications , vol.**68**, issue 8, (2008), 2261-2268.**
- W. F. Ames, Nonlinear Partial Differential Equations in Engineering, Academic Press, New York, vol. I (1965), vol. II (1972).
- G. W. Bluman and S. Kumei, Symmetries and Differential Equations, Appl. Math. Sci., 81, Springer-Verlag, New York, (1989).
- P. E. Hydon, Symmetry Methods for Differential Equations, Cambridge Texts in Applied Mathematics, Cambridge University Press, (2000).
- N.H. Ibragimov, ,,Handbook of Lie Group Analysis of Differential Equations, Volume1,2,3 CRC Press, Boca Raton, Ann Arbor, London, Tokyo, (1994,1995,1996).

- G.Baumann, Symmetry Analysis of Differential Equations with Mathematica, Telos, Springer Verlag, New York (2000).
- C. J. Budd and M. D. Piggott, Geometric integration and its applications, in Handbook of Numerical Analysis, XI, North{Holland, Amsterdam, (2003), 35-139
- A. D. Polyanin, A. I. Zhurov and A. V. Vyaz'min, Theoretical Foundations of Chemical Engineering, Vol. **34**, No. 5, (2000), 403
- S. Carstea and M.Visinescu, Mod. Phys.Lett. **A 20**, (2005), 2993-3002.
- **R.Cimpoiasu, R.Constantinescu, J.Nonlin.Math.Phys., vol** 13**, no. 2, (2006), 285-292.**
- D. Polyanin and V. F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, Chapman & Hall/CRC Press, Boca Raton, (2004), ISBN I-58488-355-3.

Thank you for attention