ON THE FINITE GROUPS WHICH ARE SPECTRUM MINIMAL, GRUNBERG-KEGEL GRAPH MINIMAL OR PRIME SPECTRUM MINIMAL

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AGREEMENT. If another isn't established, we consider finite groups only.

Recall, a group G is simple if it doesn't contain nontrivial normal subgroups.

Finite simple groups were classified. With respect to this classification, finite simple groups are:

- Alternating groups A_n for $n \geq 5$;
- Classical groups $PSL_n(q)$, $PSU_n(q)$, $PSp_n(q)$ (n is even), $P\Omega_n(q)$ (n is odd), $P\Omega_n^+(q)$ (n is even), $P\Omega_n^-(q)$ (n is even);
- Exceptional groups of Lie type: $E_8(q)$, $E_7(q)$, $E_6(q)$, ${}^2E_6(q)$, ${}^3D_n(q)$ (n is even), $F_4(q)$, ${}^2F_4(q)$, $G_2(q)$, ${}^2G_2(q)$ (q is a power of 3), ${}^2B_2(q)$ (q is a power of 2);
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Let G be a group.

DEFINITION. The spectrum $\omega(G)$ is the set of all element orders of G.

DEFINITION. The prime spectrum $\pi(G)$ is the set of all prime elements of $\omega(G)$ (equivalently, the set of all prime divisors of $|G|$).

DEFINITION. A graph $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct vertices p and q are adjacent IFF $pq \in \omega(G)$ is called the Grunberg-Kegel graph or the prime graph of G.

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Problem. What is the normal structure of a finite prime spectrum minimal group, in particular, what are its nonabelian composition factors?

Theorem 1 (M. Liebeck, C. Praeger, J. Saxl, 2000). Let G be a simple group and H is a proper subgroup of G such that $\pi(H) = \pi(G)$. Then one of the following conditions holds:
(1) $G \cong A_n$ and $A_k \leq H \leq S_k \times S_{n-k}$;
(2) $G \cong PSp_4(q)$ and $PSp_2(q^2) \trianglelefteq H$;
(3) $G \cong PSp_{2m}(q)$ where m and q are even and $\Omega^{-}_{2m}(q) \trianglelefteq H$;
(4) $G \cong P\Omega_{2m+1}(q)$ where m is even, q is odd and $\Omega^{-}_{2m}(q) \trianglelefteq H$;
(5) $G \cong P\Omega^{+}_{2m}(q)$ where m is even and $\Omega_{2m-1}(q) \trianglelefteq H$;
(6) $G \cong PSL_6(2)$ and either H is a stabilizer of a subspace of dimension 1 or 5 of the natural space of G or $H \cong PSL_5(2)$;
(7) $G \cong PSU_4(2)$ and either $H \leq 2^4.A_5$ or $H \leq S_6$;
(8) $G \cong P\Omega^{+}_8(2)$ and either $H \leq 2^6 : A_8$ or $H \leq A_9$;
(9) $G \cong PSU_4(3)$ and either $H \cong PSL_3(4)$ or $H \cong A_7$;
(10) $PSp_6(2)$ and $H \in \{S_8, A_8, S_7, A_7\}$;

(11) $G \cong G_2(3)$ and $H \cong PSL_2(13)$;
(12) $G \cong M_{11}$ and $H \cong PSL_2(11)$;
(13) $G \cong PSU_3(3)$ and $H \cong PSL_2(7)$;
(14) $G \cong PSU_3(5)$ and $H \cong A_7$;
(15) $G \cong PSU_5(2)$ and $H \cong PSL_2(11)$;
(16) $G \cong PSU_6(2)$ and $H \cong M_{22}$;
(17) $G \cong PSp_4(7)$ and $H \cong A_7$;
(18) $G \cong^2 F_4(2)'$ and $H \cong PSL_2(25)$;
(19) $G \cong M_{12}$ and either $H \cong PSL_2(11)$ or $H \cong M_{11}$;
(20) $G \cong M_{24}$ and $H \cong M_{23}$;
(21) $G \cong HS$ and $H \cong M_{22}$;
(22) $G \cong McL$ and $H \cong M_{22}$;
(23) $G \cong Co_2$ and $H \cong M_{23}$;
(24) $G \cong Co_3$ and $H \cong M_{23}$.

Theorem 2 (N.M., D. O. Revin, 2013). The group McL is not a prime spectrum minimal, but there exists a prime spectrum minimal group of form $(McL)^{104} \leftthreetimes SL_2(103)$. Every prime spectrum minimal group G containing McL as a compositional factor possesses a normal series

$$G \geq Y > Z \geq 1$$

such that

1) in the groups G/Y and Z there are not composition factors isomorphic to McL;
2) the factor-group Y/Z is a chief factor of G and is isomorphic to a direct product of some groups every of which is isomorphic to McL;
3) the factor-group G/Y is nonsolvable, it's order is not devisible by 7 and 11, and it's non-abelian composition factors are isomorphic to groups of the list: $PSL_2(q)$, $PSL_3(q)$, $PSL_4(q)$, $PSL_5(q)$, $PSU_3(q)$, $PSU_4(q)$, $PSU_5(q)$, $PSp_4(2^w)$, ${}^2B_2(2^{2m+1})$, ${}^2F_4(2^{2m+1})$, J_3.

To construct an example of a prime spectrum minimal group of form $(McL)^{104} \leftthreetimes SL_2(103)$ we used the following

Theorem 3 (A. V. Zavarnitsine, 2013). The group $\mathbb{Z}_2 \wr_\Omega PSL_2(q)$ contains a subgroup isomorphic to $SL_2(q)$ IFF $q \not\equiv 1 \pmod 4$.

Problem. Let $G \cong \mathbb{Z}_r \wr_\Omega PSL_r(q)$ where r is a prime divisor of $q-1$. When does G contain a subgroup isomorphic to $SL_r(q)$?

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(2) alternating groups A_n where n is not a prime;
(3) $PSp_4(q)$ where q odd;
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(6) $PSU_3(3)$, $PSU_4(2)$, $PSU_5(2)$, $PSp_6(2)$, $PSL_6(2)$, $G_2(3)$.

Question. Are groups $P\Omega^+_{4k}(q)$, $PSp_4(2^w)$, $PSU_3(5)$, $PSU_4(3)$, $PSU_6(2)$ isomorphic to nonabelian composition factors of finite prime spectrum minimal groups?

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Recall that the vertex set of the graph $\Gamma(G)$ of a group G is $\pi(G)$ and two distinct vertices p and q are adjacent IFF $pq \in \omega(G)$.

Question (C. Parker, 2013). Let G be a group. Is there $H < G$ such that $\Gamma(G) = \Gamma(H)$?

Example. Let $G = S_6$ and $H = S_5$. Then $\Gamma(G) = \Gamma(H)$.

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Lemma. Let $H < G$, $\Gamma(G) = \Gamma(H)$ and T be an arbitrary group. Then $H \times T < G \times T$ and $\Gamma(H \times T) = \Gamma(G \times T)$.

Remark. In view of Lemma the most interesting case of Parker's question the following

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Theorem 5 (N. M., 2014). Let G be a simple group and $H < G$. Then $\Gamma(G) = \Gamma(H)$ IFF one of the following conditions holds:

(1) (mod Golbakh's conjecture) $G \cong A_n$ where n and $n-4$ are odd composite integers and $H \cong A_{n-1}$;

(2) $G \cong PSp_4(q)$ and $H \cong PSL_2(q^2).\langle t \rangle$ where q is even and t is a field automorphism of order 2 of $PSL_2(q^2)$;

(3) $G \cong PSp_8(2^w)$ and $H \cong SO_8^-(2^w)$;

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(13) $G \cong P\Omega_8^+(2)$ and $H \in \{2^6 : A_8, A_9, S_8\}$.

Theorem was published in March 2014

N. V. Maslova, *On the coincidence of Grünberg–Kegel graphs of a finite simple group and its proper subgroup* // Trudy Instituta Matematiki i Mekhaniki UrO RAN. 20 (2014), 1, 156–168 (in Russian). English translation: Proceedings of the Steklov Institute of Mathematics (Supplementary issues). 288 (2015), suppl. 1, S129–S141.

The same result was obtained independently by Timothy C. Burness and Elisa Covato, see $arXiv: 1407.8128vl\ [math.GR]$ 30 Jul 2014

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Recall that the spectrum $\omega(G)$ is the set of all element orders of a group G.

Lemma. Let G be a group and $L \trianglelefteq H \leq G$. Then $\omega(H/L) \subseteq \omega(G)$.

Remark. If $L = 1$ and $H = G$ then $\omega(H/L) = \omega(G)$.

DEFINITION. A group G is said to be *spectrum critical* if for every subgroups L and H such that $L \trianglelefteq H \leq G$ if $\omega(H/L) = \omega(G)$ then $H = G$ and $L = 1$.

Example. Simple groups $P\Omega_8^+(2)$ and $P\Omega_8^+(3)$ aren't spectrum critical.

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Example. Simple groups $P\Omega_8^+(2)$ and $P\Omega_8^+(3)$ aren't spectrum critical.

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PROBLEM OF COINCIDENCE OF SPECTRA

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CLASSIFICATION OF FINITE SIMPLE GROUPS WHICH AREN'T SPECTRUM CRITICAL

YES!

Theorem 6 (N.M., 2014). Let G be a simple group, $L, H < G$ and $L \trianglelefteq H$. Then $\omega(G) = \omega(H/L)$ IFF $L = 1$ and one of the following conditions holds:
(1) $G \cong PSp_4(q)$ and $H \cong PSL_2(q^2).\langle t \rangle$ where q is even and t is a field automorphism of order 2 of $PSL_2(q^2)$;
(2) $G \cong PSp_8(2^w)$ and $H \cong SO_8^-(2^w)$;
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G is a prime spectrum critical group $\Rightarrow$ G is spectrum critical and prime spectrum minimal.

DEFINITION. H is a Hall subgroup of G if $(|H|, |G : H|) = 1$.

Theorem 7 (N.M., 2015). Let G be a prime spectrum minimal group. G is prime spectrum critical IFF the Fitting subgroup $F(G)$ is a Hall subgroup of G.

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Thank you for your attention!

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