On the equilibrium points of an analytic differentiable system in the plane. The center–focus problem and the divergence.

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- or finite union of elliptic, hyperbolic and parabolic sectors.

In this talk we deal with the analytic differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where the dot denotes derivative with respect to an independent real variable t. We assume that this system always is defined in a neighborhood of the origin and that the origin is a singular point.

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From now on in this talk we assume that the origin of system (1) is monodromic.

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Our aim is to show other results relating the divergence of system (1) with the solution of the center problem.

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M. GRAU AND J. LLIBRE, Divergence and Poincaré–Liapunov constants for analytic differential systems, J. Differential Equations **258** (2015), 4348–4367.

Given an analytic function $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $\mathcal{U}$ is a neighborhood of the origin, we consider its Taylor expansion at the origin:

$$f(x,y) = f_d(x,y) + \mathcal{O}_{d+1}(x,y),$$

where $d \geq 0$ is an integer and $f_d(x,y)$ is a non–zero homogeneous polynomial of degree d.

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We say that f is of sign definite if $f_d(x, y) \geq 0$ or $f_d(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$, and $f_d(x, y)$ is not identically zero.

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When $f_d(x, y) \geq 0$ (resp. $f_d(x, y) \leq 0$) for all $(x, y) \in \mathbb{R}^2$ we say that f is positive definite (resp. negative definite).

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It is clear that a necessary condition for $f(x,y)$ to be of sign definite is that d is even.

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We remark that in the case that the origin of system (1) is a strong focus (i.e. with eigenvalues $\alpha \pm \beta i$ and $\alpha \neq 0$), then the divergence $\mathrm{div}_d(x, y) = \mathrm{div}(0, 0) = 2\alpha \neq 0$ and the focus is unstable if $\mathrm{div}(0, 0) > 0$, and stable if $\mathrm{div}(0, 0) < 0$.

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PROPOSITION 1 is a generalization of this result for the strong focus to any monodromic singular point.

Assume that the origin of system (1) is a monodromic singular point, but not a strong focus.

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It is well-known that, after a linear change of variables and a constant scaling of the time variable (if necessary), the system can be written in one of the following three forms:

$$\begin{array}{llll} \dot{x} = -y + F_1(x, y), & \dot{y} = x + F_2(x, y), \\ \dot{x} = y + F_1(x, y), & \dot{y} = F_2(x, y), \\ \dot{x} = F_1(x, y), & \dot{y} = F_2(x, y), \end{array}$$

where $F_1(x, y)$ and $F_2(x, y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin.

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where $F_1(x,y)$ and $F_2(x,y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin.

These three kind of monodromic singular points are called linear type, nilpotent or degenerate, respectively.

The linear type monodromic singular points which after changes of variables can be written as

$$\dot{x} = -y + F_1(x,y), \quad \dot{y} = x + F_2(x,y),$$

are characterized by having a pair of imaginary eigenvalues.

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The degenerate monodromic singular points which after changes of variables can be written as

$$\dot{x} = F_1(x,y), \quad \dot{y} = F_2(x,y),$$

can be characterized using blow–ups.

Assume that we have the system

$$\begin{aligned} \dot{x} &= P(x,y) = P_n(x,y) + O_{n+1}(x,y), \\ \dot{y} &= Q(x,y) = P_m(x,y) + O_{m+1}(x,y), \end{aligned} \tag{2}$$

where $n \geq 1$ and $m \geq 1$ are integers and $P_n(x,y)$ and $Q_m(x,y)$ are non–zero homogeneous polynomials of degrees n and m respectively, formed by the lowest order terms of $P(x,y)$ and $Q(x,y)$, respectively.

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Define the real polynomial

$$\Delta(x,y) = \begin{cases} yP_n(x,y) - xQ_m(x,y) & \text{if } n = m, \\ yP_n(x,y) & \text{if } n < m, \\ -xQ_m(x,y) & \text{if } n > m. \end{cases}$$

A sufficient condition in order that system (2) has a monodromic singular point at the origin is that $\Delta(x, y) = 0$ only if $(x, y) = (0, 0)$.

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A necessary condition in order that system (2) has a monodromic singular point at the origin is that $\Delta(x, y)$ is of sign definite.

Suppose that the origin is a monodromic singular point. Then we have the Poincaré map $\mathcal{P} : [0, x^*) \to [0, \infty)$, being $\mathcal{P}(x)$ the point in $[0, \infty)$ corresponding to the first cut with $[0, \infty)$ of the orbit through the point $(x, 0)$ in positive time.

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It is clear that the origin of system (1) is a center if and only if this Poincaré map is the identity.

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For linear type singular points always the Poincaré map is analytic at $x = 0$ and writes as

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Also for degenerate monodromic singular points having no characteristic directions the Poincaré map is analytic at $x = 0$.

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We define the $2k+1$ Poincaré–Liapunov constant as the expression α_{2k+1} modulus the vanishing of all the previous ones.

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If $\alpha_{d+1} \neq 0$, then it is the non–zero first Poincaré–Liapunov constant, and consequently the origin is a focus.

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where $P_s(x, y)$ and $Q_s(x, y)$ are homogeneous polynomials of odd degree s. Then the first Poincaré–Liapunov constants of system (1) are $\alpha_i = 0$ for $i = 1, 2, \ldots, s-1$ and

$$\alpha_s = \frac{1}{s+1} \int_0^{2\pi} \mathrm{div}(\cos t, \sin t)\, dt.$$

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$$V_{d+1}(\varepsilon) = \int_0^{2\pi/\sqrt{\varepsilon}} \mathrm{div}_d \left(\cos(\sqrt{\varepsilon}\, t), \; -\sqrt{\varepsilon} \sin(\sqrt{\varepsilon}\, t)\right) \, dt,$$

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where $\varepsilon > 0$, and define the constant v_{d+1} through the series expansion $V_{d+1}(\varepsilon) = \dfrac{v_{d+1}}{\sqrt{\varepsilon}} + O(\varepsilon)$.

(a) If the origin is a center, then $v_{d+1} = 0$ for all $\varepsilon > 0$.

(b) If $v_{d+1} > 0$ (resp. $v_{d+1} < 0$), then the origin is an unstable (resp. stable) focus.

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In this case the degree of the lowest order terms of $P(x,y)$ and $Q(x,y)$ must coincide, that is,

$$P(x,y) = P_n(x,y) + \mathcal{O}_{n+1}(x,y),$$
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We define

$$v(\theta) = \exp\left[\int_0^\theta \frac{\cos\sigma P_n(\cos\sigma, \sin\sigma) + \sin\sigma Q_n(\cos\sigma, \sin\sigma)}{\cos\sigma Q_n(\cos\sigma, \sin\sigma) - \sin\sigma P_n(\cos\sigma, \sin\sigma)} \, d\sigma\right].$$

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$$\alpha = \int_0^{2\pi} \frac{\mathrm{div}_d\,(\cos\theta, \sin\theta)\ \, v(\theta)^{d-n+1}}{\cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta)}\, d\theta \neq 0.$$

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Then the origin is a focus which is stable (resp. unstable) if $\alpha < 0$ (resp. $\alpha > 0$).

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$$\alpha = \int_0^{2\pi} \frac{\mathrm{div}_d(\cos\theta, \sin\theta)\ v(\theta)^{d-n+1}}{\cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta)}\, d\theta \neq 0.$$

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We remark that from PROPOSITION 1, if $v(2\pi) > 1$ then the origin is an unstable focus, and if $v(2\pi) < 1$ then the origin is a stable focus.

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We remark that from PROPOSITION 1, if $v(2\pi) > 1$ then the origin is an unstable focus, and if $v(2\pi) < 1$ then the origin is a stable focus. So this theorem is useful to establish the stability of the origin when $v(2\pi) = 1$.

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PROPOSITION 1 Assume that the origin of an analytic differential system (1) is a monodromic singular point, and that the divergence $\mathrm{div}(x, y)$ of system (1) is of sign definite.

Then the origin of system (1) is a focus; either unstable if the divergence is positive definite or stable if it is negative definite.

Proof of PROPOSITION 1.

The Bendixson criterium: If the divergence of a system (1) is not identically zero and does not change sign in a simply connected region in $\mathbb{R}^2$, then there is no closed orbit lying entirely in this simply connected region.

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If the divergence of system (1) is of sign definite, then there is a neighborhood $\mathcal{U}_O$ of the origin in which $\text{div}(x, y) \geq 0$ or $\text{div}(x, y) \leq 0$ for all $(x, y) \in \mathcal{U}_O$.

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If the origin is a center, then there is a continuum of periodic orbits completely contained in $\mathcal{U}_O$ which contradicts the Bendixson criterium.

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Now it remains to study the kind of stability this focus.

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We consider a transversal section Σ whose boundary contains the origin O and a neighborhood $\mathcal{U}_O$ of the origin such that $\mathrm{div}(x, y) \geq 0$ for all $(x, y) \in \mathcal{U}_O$. We only consider the part of Σ contained in $\mathcal{U}_O$.

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We fix a point ρ in Σ and we consider the point in Σ corresponding to its image by the Poincaré map $\mathcal{P}(\rho)$ (when this is defined).

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We fix a point ρ in Σ and we consider the point in Σ corresponding to its image by the Poincaré map $\mathcal{P}(\rho)$ (when this is defined). If ρ is close enough to the origin, we can ensure that $\mathcal{P}(\rho)$ is contained in $\mathcal{U}_O$.

We define the closed curve C formed by the arc of the orbit from ρ to $\mathcal{P}(\rho)$ together with the arc of Σ between these two points.

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Since Σ is a transversal section, we have that all the orbits of (1) cross ℓ in the same direction, either inside or outside the region D limited by the curve C and the segment ℓ.

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The origin is stable if the orbits cross ℓ in the inside direction and unstable otherwise.

We consider

$$\oint_C Pdy - Qdx = \int_{C\setminus\ell} Pdy - Qdx + \int_\ell Pdy - Qdx.$$

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On the other hand, since $\mathrm{div}(x, y) \geq 0$ for all $(x, y) \in D$, we have by the Green's formula

$$\oint_C Pdy - Qdx = \iint_D \mathrm{div}(x, y)\, dxdy > 0.$$

So

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This implies that all the orbits of (1) cross ℓ in the outside direction and, thus, the origin of (1) is unstable.

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This implies that all the orbits of (1) cross ℓ in the outside direction and, thus, the origin of (1) is unstable. This completes the proof of PROPOSITION 1.

THEOREM 2 Consider an analytic differential system (1) whose origin is a linear type monodromic singular point. Denote by $\mathrm{div}_d(x, y)$ the lowest order terms of the divergence $\mathrm{div}(x, y)$ of the system. Define

$$\alpha_{d+1} = \frac{1}{d+2} \int_0^{2\pi} \mathrm{div}_d\,(\cos t, \sin t)\; dt.$$

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Its proof uses the Birkhoff normal form of a center provided in

G. BELITSKIĬ, Smooth equivalence of germs of vector fields with one zero or a pair of purely imaginary eigenvalues, Funct. Anal. Appl. **20** (1986), 253–259.

THEOREM 3 Consider an analytic differential system (1) whose origin is a nilpotent monodromic singular point. Denote by $\mathrm{div}_d(x, y)$ the lowest order terms of the divergence $\mathrm{div}(x, y)$ of the system. Define

$$V_{d+1}(\varepsilon) = \int_0^{2\pi/\sqrt{\varepsilon}} \mathrm{div}_d \left(\cos(\sqrt{\varepsilon}\, t), \; -\sqrt{\varepsilon} \sin(\sqrt{\varepsilon}\, t)\right) \, dt,$$

where $\varepsilon > 0$, and define the constant v_{d+1} through the series expansion $V_{d+1}(\varepsilon) = \dfrac{v_{d+1}}{\sqrt{\varepsilon}} + O(\varepsilon)$.

(a) If the origin is a center, then $v_{d+1} = 0$ for all $\varepsilon > 0$.

(b) If $v_{d+1} > 0$ (resp. $v_{d+1} < 0$), then the origin is an unstable (resp. stable) focus.

The proof of Theorem 3 uses results from

J. GINÉ AND J. LLIBRE, A method for characterizing nilpotent centers, J. Math. Anal. Appl. **413** (2014), 537–545.

I.A. GARCÍA, H. GIACOMINI, J. GINÉ AND J. LLIBRE, Analytic nilpotent centers as limits of nondegenerated centers revisited, Preprint.

THEOREM 4 Consider an analytic differential system (1) whose origin is monodromic and has no characteristic directions. Denote by $\mathrm{div}_d(x, y)$ the lowest order terms of degree d of the divergence $\mathrm{div}(x, y)$ of the system. Assume that $v(2\pi) = 1$ and

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The proof follows by direct computations.

For more details on the proofs of THEOREMS 2, 3 and 4 see the paper:

M. GRAU AND J. LLIBRE, Divergence and Poincaré–Liapunov constants for analytic differential systems, J. Differential Equations **258** (2015), 4348–4367.

THANK YOU VERY MUCH FOR YOUR ATTENTION