Applications of Commuting graphs

B. Kuzma

University of Primorska, Koper

Institute of Mathematics, Physics and Mechanics, Ljubljana

Commuting graphs

$\mathcal{A}$ a magma, $Z(\mathcal{A}) = \{x \in \mathcal{A};\ \ ax = xa\ \forall a \in \mathcal{A}\}$.

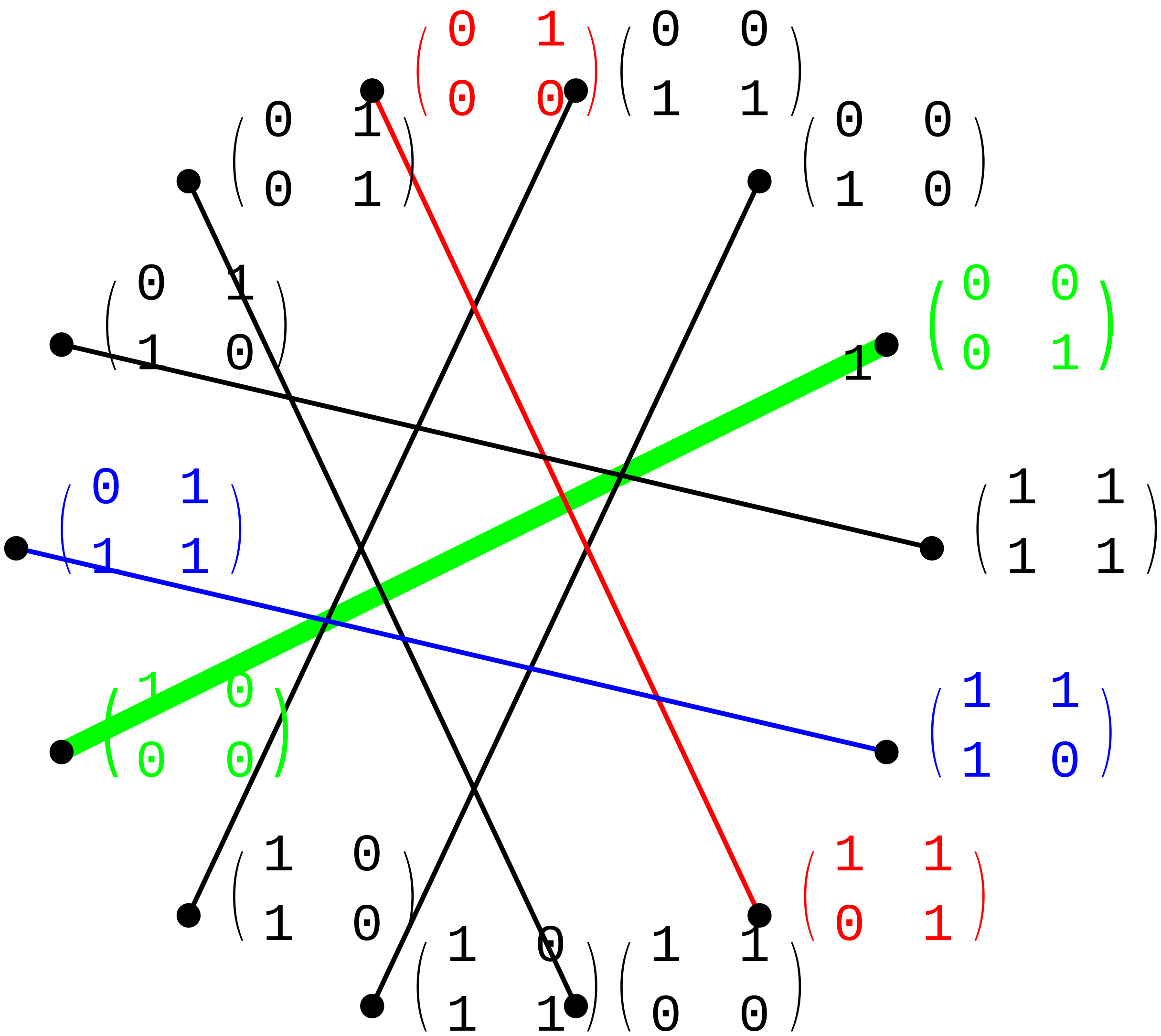
Its *commuting graph*, $\Gamma = \Gamma(\mathcal{A})$, is simple graph with

$$V(\Gamma) = \mathcal{A} \setminus Z(\mathcal{A}); \qquad X - Y \text{ iff } \begin{cases} XY = YX \\ X \neq Y \end{cases}.$$

NOT EASY TO VISUALIZE!

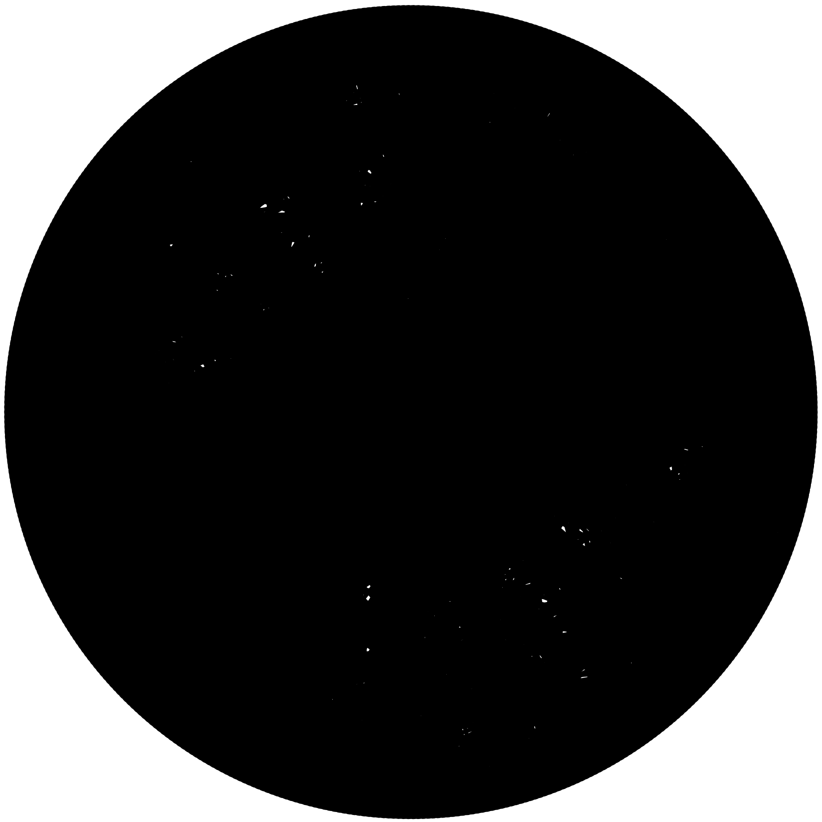
Commuting graphs: $\Gamma(M_2(Z_2))$

- $V(\Gamma) = \mathcal{A} \setminus \mathcal{Z}(\mathcal{A})$; $\quad E(\Gamma) = \{(a, b);\ \ ab = ba,\ a \neq b\}$.

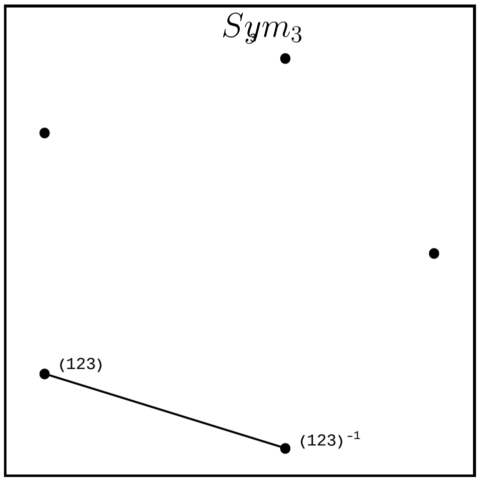


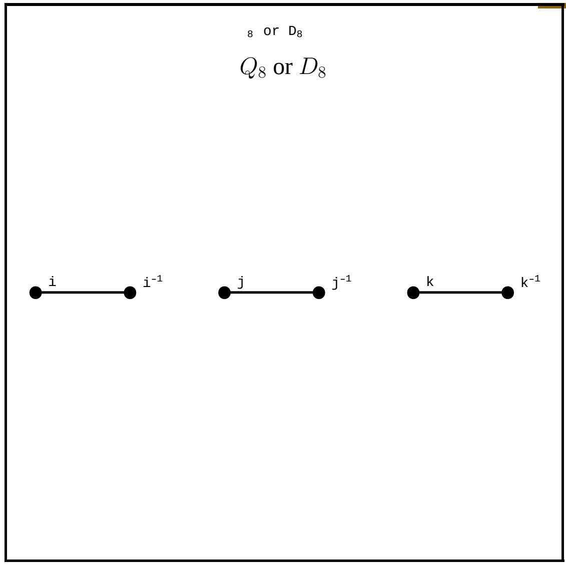
Commuting graphs: $\Gamma(M_3(Z_2))$

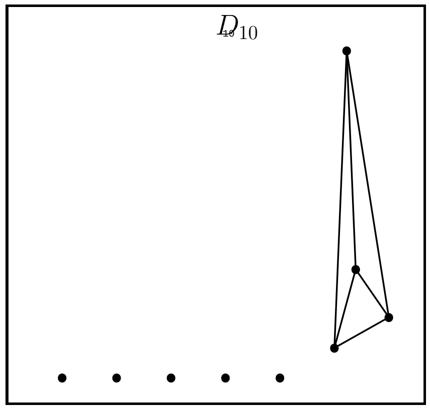
- $V(\Gamma) = \mathcal{A} \setminus \mathcal{Z}(\mathcal{A})$; $\quad E(\Gamma) = \{(a, b);\ ab = ba,\ a \neq b\}$.

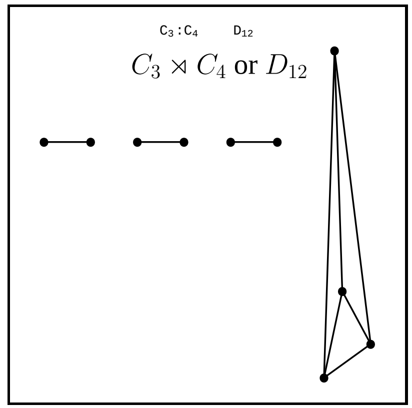


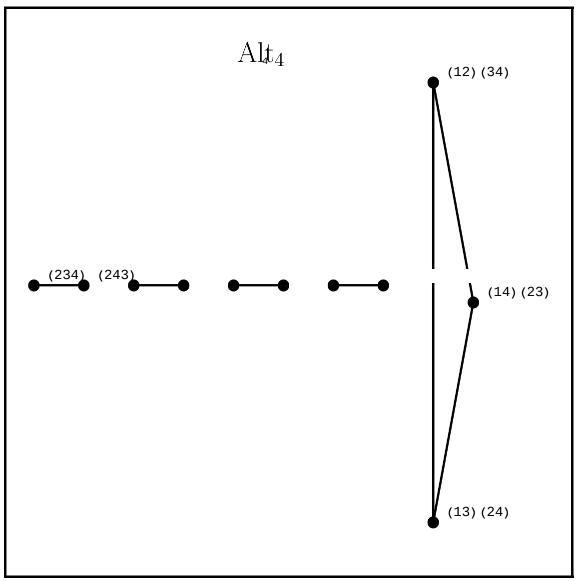
Commuting graphs: Groups!

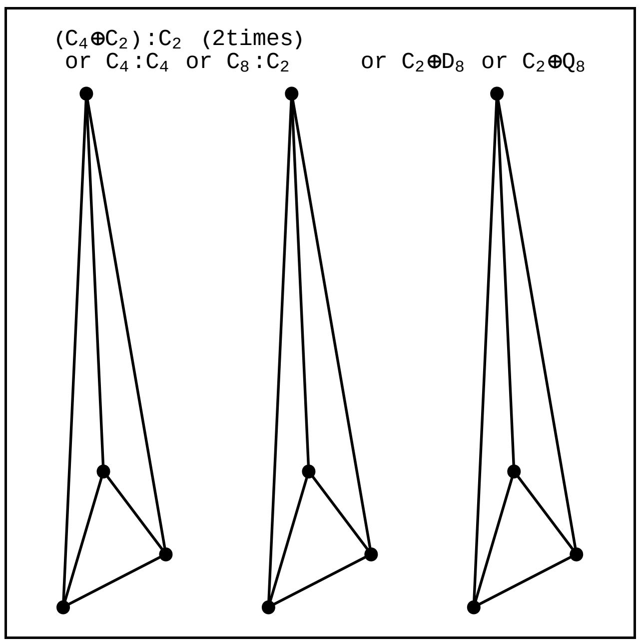


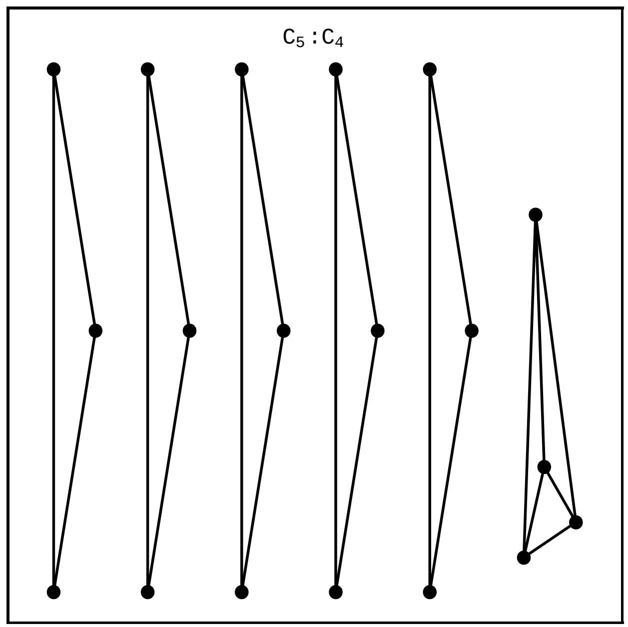




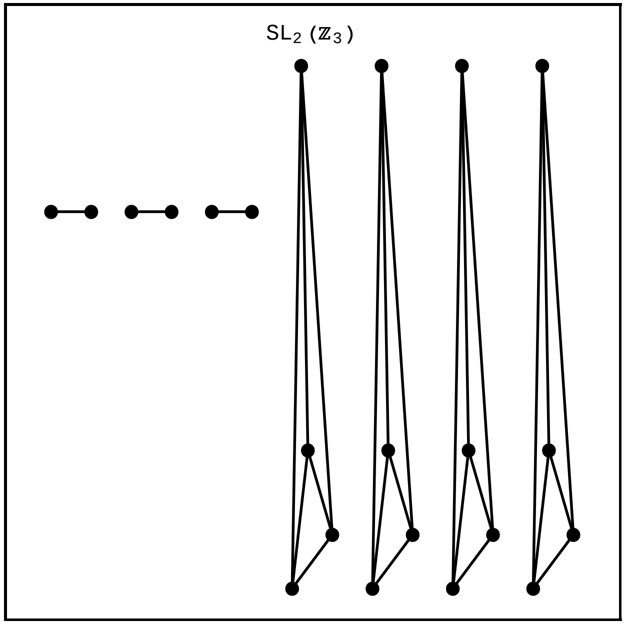


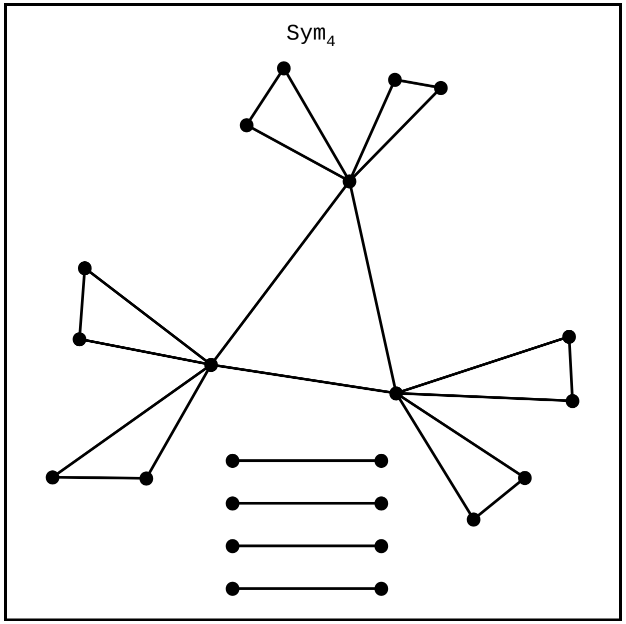


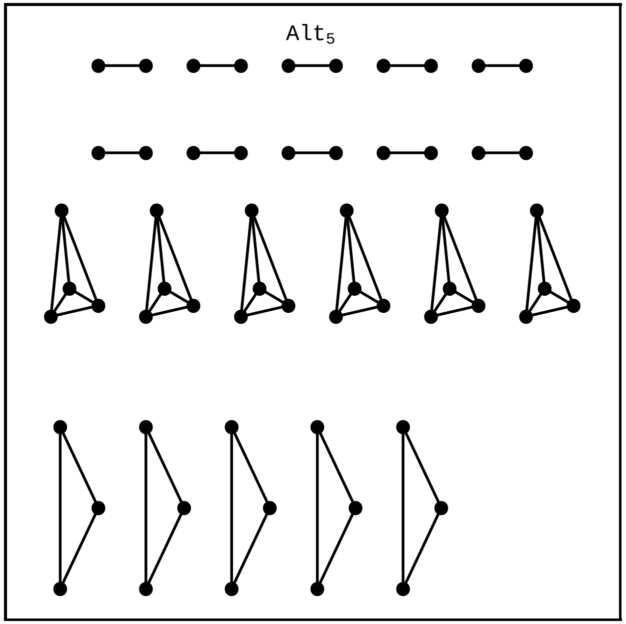




Commuting graphs: Groups!







Fundamental question for Γ.

APPLICATIONS!?

Answer: In Applications!

PHILOSOPHY: Often, a given algebra has "a lot of (non)commuting pairs".

Theorem (Watkins (1976))**.** *A linear bijection* $\phi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$, *(*$n \geq 4$*) preserves commutativity. THEN*

$$\begin{cases} \phi(X) = cTXT^{-1} + f(X)I \\ \phi(X) = \left(cTXT^{-1} + f(X)I\right)^t \end{cases}$$

Answer: In Applications!

Theorem (Dolinar, K., submitted)**.**
$\phi\colon M_n(\mathbb{C}) \xrightarrow{\text{surjective}} M_n(\mathbb{C})$, preserves commutativity.
Assume $\phi(X) \in \mathbb{C}I$ implies $X \in \mathbb{C}I$.

THEN: ϕ is surjective homomorphism of commuting graph. Moreover,

$$\begin{cases} \phi(R) = \gamma_R T^{-1} R_\sigma T \\ \phi(R) = \gamma_R T^{-1} (R_\sigma)^t T \end{cases} \quad ; \qquad \mathrm{rank} R = 1$$

$\gamma_R \in \mathbb{C}$, $R_\sigma := \big(\sigma(r_{ij})\big)_{ij}$ where $\mathbb{C} \xrightarrow{\sigma} \mathbb{C}$ field isomorphism.

Answer: In Applications!

- **Theorem** (Mohammedian 2010).
  $\mathbb{F} = GF(p^k)$ a finite field, R a unital ring.

  IF $\Gamma(R) \sim \Gamma(M_2(\mathbb{F}))$ THEN $R \sim M_2(\mathbb{F})$.

Answer: In Applications!

- **Theorem** (Mohammedian 2010).
  $\mathbb{F} = GF(p^k)$ a finite field, R a unital ring.

  IF $\Gamma(R) \sim \Gamma(M_2(\mathbb{F}))$ THEN $R \sim M_2(\mathbb{F})$.
- **Theorem** (Solomon, Woldar 2014).
  S a finite, simple, nonabelian group; G a group.

  IF $\Gamma(S) \sim \Gamma(G)$ THEN $S \sim G$.

Basic problems for Γ

- Homomorphisms (=commutativity preserving maps without linearity).
- Isomorphism problem.

Basic problems for Γ

- Homomorphisms (=commutativity preserving maps without linearity).
- Isomorphism problem.
- Diameter/connectedness problem.
- Realization problem.
- Structure recognition problem.

Commuting graph of $\mathscr{B}(\mathcal{H})$

NOTATIONS:

- $\mathcal{H}$ a complex Hilbert space, $\dim \mathcal{H} \le \infty$.
- $\mathscr{B}(\mathcal{H})$ Banach algebra of bounded operators on $\mathcal{H}$.
- $\Gamma = \Gamma(\mathscr{B}(\mathcal{H}))$.
- $A' := \{X \in \mathscr{B}(\mathcal{H}); \;\; AX = XA\}$ a commutant.
- $A — B — C — \ldots$ (a path in Γ)

  means $(AB - BA) = 0 = (BC - CB) = \ldots$.

- $E_{ij} \in M_n(\mathbb{C})$ a STD matrix unit.

Properties

- $\dim \mathcal{H} = 2$. THEN: $\Gamma(M_2(\mathbb{C}))$ is not connected.

PROOF

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}' = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, and $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}' = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.

So, $E_{11} — \ldots — E_{12}$ does not exists.

Properties

- $\dim \mathcal{H} = 2$. THEN: $\Gamma(M_2(\mathbb{C}))$ is not connected.

PROOF

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}' = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, and $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}' = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.
So, $E_{11} — \ldots — E_{12}$ does not exists.

Actually, $\Gamma(M_2(\mathbb{C})) = \infty K_\infty$!

Properties

- $\dim \mathcal{H} = 2$. THEN: $\Gamma(M_2(\mathbb{C}))$ is not connected.
- (Akbari-Mohammadian-Radjavi-Raja '06) $\dim \mathcal{H} = n \geq 3$. THEN: $\Gamma(M_n(\mathbb{C}))$ always connected with diameter 4.

PARTIAL PROOF

A path of length four between A, B:

- A has e.vector x (corresponding to e.value λ).
- A^{tr} has e.vector f (again to e.value λ).
- Hence, $A — (xf^{\mathrm{tr}})$. Likewise exists y, g with $(yg^{\mathrm{tr}}) — B$.
- $\exists$ nonzero z, h with $f^{\mathrm{tr}} z = 0 = g^{\mathrm{tr}} z$ and $h^{\mathrm{tr}} x = 0 = h^{\mathrm{tr}} y$.
  THEN, $A — (xf^{\mathrm{tr}}) — (zh^{\mathrm{tr}}) — (yg^{\mathrm{tr}}) — B$.
- Can show $d(J^{\mathrm{tr}}, J) = 4$.

Properties

- $\dim \mathcal{H} = 2$. THEN: $\Gamma(M_2(\mathbb{C}))$ is not connected.
- (Akbari-Mohammadian-Radjavi-Raja '06) $\dim \mathcal{H} = n \geq 3$. THEN: $\Gamma(M_n(\mathbb{C}))$ always connected with diameter 4.

- If $\mathbb{F} \neq \bar{\mathbb{F}}$,
  THEN: A, B may lack e.vectors, hence proof fails!

  Worse still: commuting graph may be disconnected.

Commuting graph of $\mathscr{B}(\mathcal{H})$

Theorem (Ambrozie-Bračič-Müller-K.**).** *If* $\mathcal{H}$ *is non-separable,* *THEN* $\operatorname{diam}(\Gamma) = 2$.

PROOF

- Choose $A, B \in \mathscr{B}(\mathcal{H}) \backslash \mathbb{C}I$.
- Define $\mathcal{W} := \mathrm{Semigp}\{I, A, A^*, B, B^*\}$, fix nonzero $x \in \mathcal{H}$.
- $\mathcal{N} := \bigvee \mathcal{W}x$ is closed, separable subspace, and contains x.
- HENCE: $\mathcal{N}$ is a proper reducing subspace for A and B.
- Let P be orthogonal projection on $\mathcal{N}$.
- Since $\mathcal{N}$ is reducing, $A — P — B$. $\square$

Commuting graph of $\mathscr{B}(\mathcal{H})$

Theorem (Ambrozie-Bračič-Müller-K.). *If* $\mathcal{H} = \ell^2$ *is separable, THEN* $\mathrm{diam}(\Gamma) = \infty$.
Moreover, $\exists T \in \mathscr{B}(\mathcal{H})$ *such that*

$$T' = X' \qquad X \in T' \backslash \mathbb{C}I.$$

Commuting graph of $\mathscr{B}(\mathcal{H})$

Theorem (Ambrozie-Bračič-Müller-K.). $\dim \mathcal{H} = \infty$. *THEN,*

✓ *Finite rank operators,*
✓ *operators with disconnected spectrum,*
✓ *nonscalar operators similar to*
(i) normal or (ii) to $\mathcal{C}_0$*-contractions or*
(iii) to weighed shifts or (iv) to partial isometries

are in the same connected component of $\Gamma(\mathscr{B}(\mathcal{H}))$.

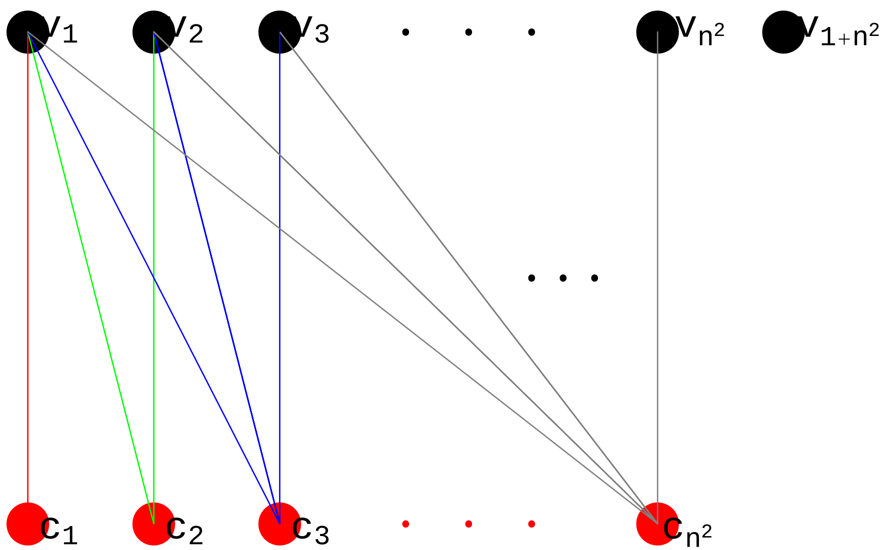
Realizability problem

Theorem (Realizability, Ambrozie,Bračič,K., Müller)**.** *Let* Γ *be a simple graph. Then,* Γ *is isomorphic to a commuting subgraph of* $\mathscr{B}(\mathcal{H})$*, spanned by rank-two projections. If* Γ *is finite, then* $\dim \mathcal{H} < \infty$.

Remark. Similar question was considered by T. Pisanski for realizabilitry of finite graphs as commuting graphs of GROUPS.

Realizability problem

Theorem (Nonrealizability Ambrozie,Bračič,K., Müller**).** *The graph on* $|\Gamma| = 2n^2 + 1$ *vertices which cannot be embedded as a commuting graph of* $M_n(\mathbb{C})$.



Property recognition

Theorem (Dolinar,Oblak,K.)**.**
$n \geq 3$. *TFAE for* $B \in M_n(\mathbb{C})$.

(i) B *nonderogatory.*

(ii) B *is minimal* *(i.e.* $X' \subseteq B'$ *implies* $X' = B'$*)*.

(iii) $\exists X$ *with* $d(B, X) = 4$.

Property recognition

Theorem (Dolinar,Oblak,K.)**.**
$n \geq 3$. *TFAE for* $B \in M_n(\mathbb{C})$.

(i) B *nonderogatory.*

(ii) B *is minimal* (i.e. $X' \subseteq B'$ *implies* $X' = B'$)**.**

(iii) $\exists X$ *with* $d(B, X) = 4$.

- $(i) \iff (ii)$ by Šemrl.
- $\neg(i) \implies \neg(iii)$ IDEA. Fix X, can find rank-one R with $RX = XR$. Assume

$B = \left(\begin{array}{c|c} J_{n_1} & \\ \hline & J_{n_2} \end{array}\right)$. Then, $Z = \left(\begin{array}{c|c} J_{n_1} \; x_1 & x_2 \\ \hline x_3 & J_{n_2} \; x_4 \end{array}\right)$ satisfies $BZ = ZB$. If $R \in M_{n_1+n_2}(\mathbb{F})$ is any rank-one, can find nonzero x_i so that $RZ = ZR$. Hence, $B — Z — R — X$.

Property recognition

Theorem (Dolinar,Oblak,K.)**.**
$n \geq 3$. *TFAE for* $B \in M_n(\mathbb{C})$.

(i) B *nonderogatory.*

(ii) B *is minimal* *(i.e.* $X' \subseteq B'$ *implies* $X' = B'$*)*.

(iii) $\exists X$ *with* $d(B, X) = 4$.

- $(i) \implies (iii)$ WLOG $\quad B = \bigoplus_{j=1}^{\ell} J_{m_j}(\mu_j)$
in upper-triangular Jordan form. Adapting the proof of A.M.R.R., can show: $d(B, J^{\mathrm{tr}}) = 4$.

Property recognition

Theorem (Dolinar,Oblak,K.).
$n \geq 3$. *TFAE for* $B \in M_n(\mathbb{C})$.

(i) B *nonderogatory.*

(ii) B *is minimal* *(i.e.* $X' \subseteq B'$ *implies* $X' = B'$*).*

(iii) $\exists X$ *with* $d(B, X) = 4$.

Actually, $(ii) \implies (iii)$ follows from a more general fact:

Theorem (D.O.K.). $A = \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i), \quad B = \bigoplus_{j=1}^{\ell} J_{m_j}(\mu_j)$ *nonderogatory of any given type* $n_1 + \cdots + n_k = n = m_1 + \cdots + m_\ell$. *THEN,*

$$d(A, S^{-1}BS) = 4; \qquad \left(S = \left(\frac{1}{x_i - y_j} \right)_{ij} \right).$$

Property recognition

Minimal matrices are the ones that come at maximal distance.

Theorem (Classification of rank-one)**.** *TFAE for nonscalar* A.

(i) $A' = R'$ *for some rank-one* R.

(ii) $d(A, X) \leq 2$ *for every nonminimal* X.

Theorem (Classification of semisimplicity (=diagonalizability))**.**
TFAE for nonscalar A.

(i) A *is semisimple*

(ii) $\exists$ *minimal* $B — A$ *such that for any* $Y — X — B$ *can find minimal* M *with* $Y — M — X$.

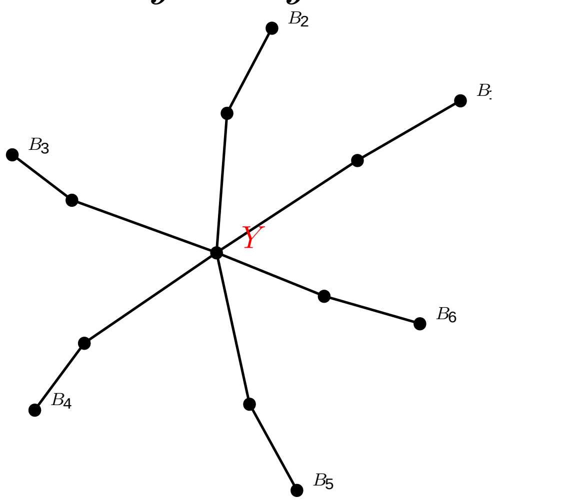
Property recognition

Lemma (Dolinar-K., submitted). *TFAE for* $A \in M_n(\mathbb{C}) \backslash \mathbb{C}I$, *for* $n \geq 5$:

(i) $A' = R'$ *for some* $\mathrm{rank} R = 1$.

(ii) $\forall\ (n-2)$*-tuple* $B_1, \ldots, B_{n-2}$ *with* $d(B_i, B_j) = 4$, *(*$i \neq j$*)*
exists $Y \in A' \backslash \mathbb{C}I$ *and nonscalar matrices* X_{ij}, Z_{ij} *such that*

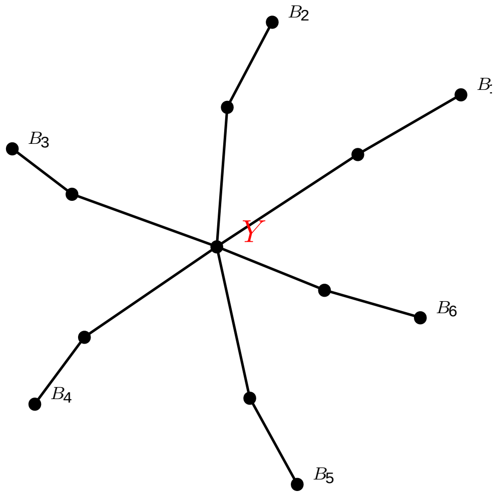
$$B_i — X_{ij} — Y — Z_{ij} — B_j.$$



Property recognition

(ii) $\forall\ (n-2)$-tuple $B_1, \ldots, B_{n-2}$ with $d(B_i, B_j) = 4$, $(i \neq j)$
exists $Y \in A' \backslash \mathbb{C}I$ and nonscalar matrices X_{ij}, Z_{ij} such that

$$B_i — X_{ij} — Y — Z_{ij} — B_j.$$



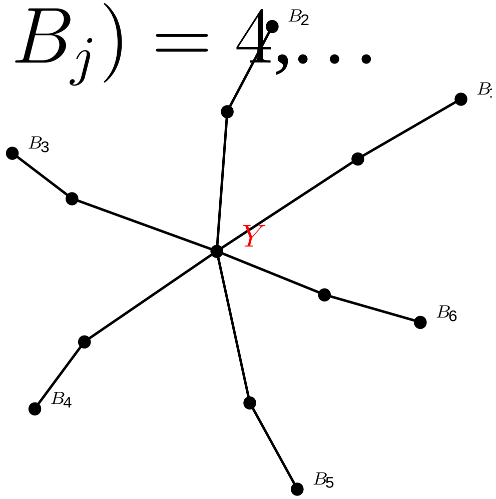
PARAPHRASING:

$\exists A \in M_n(\mathbb{C}) \backslash \mathbb{C}I$ such that (ii) holds for every $(n-2)$ tuple $B_1, \ldots, B_{n-2}$ with $d(B_i, B_j) = 4$ $(i \neq j)$.

Property recognition

(ii) $\forall\ (n-2)$-tuple $B_1, \ldots, B_{n-2}$ with $d(B_i, B_j) = 4, \ldots$

$$B_i — X_{ij} — {\color{red}Y} — Z_{ij} — B_j.$$



PARAPHRASING:

$\exists A \in M_n(\mathbb{C}) \backslash \mathbb{C}I$ such that (ii) holds for every $(n-2)$ tuple $B_1, \ldots, B_{n-2}$ with $d(B_i, B_j) = 4$ ($i \neq j$).

HOWEVER: $\forall A \in M_n(\mathbb{C}) \backslash \mathbb{C}I$ exists a $(n-1)$-tuple $B_1, \ldots, B_{n-1}$ without "star-shaped" path through nonscalar ${\color{red}Y} \in A'$.

Back to isomorphism problem

Thus, $\Gamma(M_n(\mathbb{C}))$ and $\Gamma(M_m(\mathbb{C}))$ are not isomorphic if $n \neq m$.

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.

(and hence also $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$*).*

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.

(*and hence also* $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$).

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.

(*and hence also* $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$***).***

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.

(*and hence also* $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$).

PROBLEM:
Let $\mathcal{A}$ be a prime $\mathcal{C}^*$-algebra with $\Gamma(\mathcal{A}) \sim \Gamma(\mathscr{B}(\mathcal{H}))$ for some $\mathcal{H}$.

Does it follow that $\mathcal{A} \sim \mathscr{B}(\mathcal{H})$?

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.
(and hence also $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$*).*

PROBLEM:
Let $\mathcal{A}$ be a prime $\mathcal{C}^*$-algebra with $\Gamma(\mathcal{A}) \sim \Gamma(\mathscr{B}(\mathcal{H}))$ for some $\mathcal{H}$.

Does it follow that $\mathcal{A} \sim \mathscr{B}(\mathcal{H})$?

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.
(*and hence also* $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$).

PROBLEM:
Let $\mathcal{A}$ be a prime $\mathcal{C}^*$-algebra with $\Gamma(\mathcal{A}) \sim \Gamma(\mathscr{B}(\mathcal{H}))$ for some $\mathcal{H}$.

Does it follow that $\mathcal{A} \sim \mathscr{B}(\mathcal{H})$?

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.
(and hence also $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$*).*

PROBLEM:
Let $\mathcal{A}$ be a prime $\mathcal{C}^*$-algebra with $\Gamma(\mathcal{A}) \sim \Gamma(\mathscr{B}(\mathcal{H}))$ for some $\mathcal{H}$.

Does it follow that $\mathcal{A} \sim \mathscr{B}(\mathcal{H})$?

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.

(and hence also $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$*).*

PROBLEM:
Let $\mathcal{A}$ be a prime $\mathcal{C}^*$-algebra with $\Gamma(\mathcal{A}) \sim \Gamma(\mathscr{B}(\mathcal{H}))$ for some $\mathcal{H}$.

Does it follow that $\mathcal{A} \sim \mathscr{B}(\mathcal{H})$?

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.
(*and hence also* $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$).

PROBLEM:
Let $\mathcal{A}$ be a prime $\mathcal{C}^*$-algebra with $\Gamma(\mathcal{A}) \sim \Gamma(\mathscr{B}(\mathcal{H}))$ for some $\mathcal{H}$.

Does it follow that $\mathcal{A} \sim \mathscr{B}(\mathcal{H})$?

Back to isomorphism problem

Corollary. *If* $\Gamma(\mathscr{B}(\mathcal{H})) \sim \Gamma(\mathscr{B}(\mathcal{K}))$ *then* $\dim \mathcal{H} = \dim \mathcal{K}$.
(and hence also $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$*).*

PROBLEM:
Let $\mathcal{A}$ be a prime $\mathcal{C}^*$-algebra with $\Gamma(\mathcal{A}) \sim \Gamma(\mathscr{B}(\mathcal{H}))$ for some $\mathcal{H}$.

Does it follow that $\mathcal{A} \sim \mathscr{B}(\mathcal{H})$?

Property recognition again.

- $B \in M_n(\mathbb{F})$ given. Neighborhood of B is

$$\{X \in \Gamma(M_n(\mathbb{F}));\ \ d(B, X) = 1\}$$
$$= \mathcal{C}(B) \setminus (\{B\} \cup \mathbb{F}\ \mathrm{Id}).$$

Much known of $\mathcal{C}(B)$!

Property recognition again.

- $B \in M_n(\mathbb{F})$ given. Neighborhood of B is

$$\{X \in \Gamma(M_n(\mathbb{F}));\ \ d(B, X) = 1\}$$
$$= \mathcal{C}(B) \setminus (\{B\} \cup \mathbb{F}\ \mathrm{Id}).$$

  Much known of $\mathcal{C}(B)$!

- The other extreme: When $d(B, X) = \max$?

Commuting graph-new results

Theorem 2 (Dolinar-Oblak-K.)**.**
$\mathbb{F} = \bar{\mathbb{F}}$ *and* $n \geq 3$. *TFAE for* $B \in M_n(\mathbb{F})$.

(i) B *nonderogatory.*

(ii) B *is minimal* (*i.e.* $\mathcal{C}(X) \subseteq \mathcal{C}(B)$ *implies* $\mathcal{C}(X) = \mathcal{C}(B)$)**.**

(iii) $\exists X$ *with* $d(B, X) = 4$.

Commuting graph-new results

Theorem 2 (Dolinar-Oblak-K.)**.**
$\mathbb{F} = \bar{\mathbb{F}}$ *and* $n \geq 3$. *TFAE for* $B \in M_n(\mathbb{F})$.

(i) B *nonderogatory.*

(ii) B *is minimal* *(i.e.* $\mathcal{C}(X) \subseteq \mathcal{C}(B)$ *implies* $\mathcal{C}(X) = \mathcal{C}(B)$*)*.

(iii) $\exists X$ *with* $d(B, X) = 4$.

- $(i) \implies (iii)$

B nonderogatory, so possess cyclic vector. Hence:
WLOG $\quad B = C(f)$.
Adapting the proof of A.M.R.R., can show: $d(B, J) = 4$.

Commuting graph-new results

Theorem 2 (Dolinar-Oblak-K.**).**
$\mathbb{F} = \bar{\mathbb{F}}$ and $n \geq 3$. TFAE for $B \in M_n(\mathbb{F})$.

(i) B *nonderogatory.*

(ii) B *is minimal (i.e. $\mathcal{C}(X) \subseteq \mathcal{C}(B)$ implies $\mathcal{C}(X) = \mathcal{C}(B)$).*

(iii) $\exists X$ *with* $d(B, X) = 4$.

- $\neg(i) \implies \neg(iii)$ IDEA. Fix X, can find rank-one R with $RX = XR$. Assume $B = \left(\begin{array}{c|c} J_{n_1} & \\ \hline & J_{n_2} \end{array}\right)$. Then, $Z = \left(\begin{array}{c|c} x_1 & x_2 \\ \hline x_3 & x_4 \end{array}\right)$ satisfies $BZ = ZB$. If $R \in M_{n_1+n_2}(\mathbb{F})$ is any rank-one, can find nonzero x_i so that $RZ = ZR$. Hence, $B — Z — R — X$.

Commuting graph-new results

Nonderogatory matrices are the ones that come at maximal distance.

Theorem 3 (Classification of rank-one). *TFAE for nonscalar A.*

(i) A *is* $\mathcal{C}$*-equivalent to rank-one* (*i.e.* $A' = R'$ *for some rank-one* R).

(ii) $d(A, X) \leq 2$ *for every derogatory* X.

Theorem 4 (Classification of semisimplicity (=diagonalizability)).
TFAE for nonscalar A.

(i) A *is semisimple*

(ii) $\exists$ *nonderog.* $B — A$ *such that for any* $Y — X — B$ *can find nonderog.* M *with* $Y — M — X$.

[1] C. Ambrozie, J. Bračič, V. Müler, and B. Kuzma, *The commuting graph of bounded linear operators on a Hilbert space*. J. Funct. Anal. **264** (2013) 1068–1087.
[2] A. Mohammadian, *On Commuting Graphs of Finite Matrix Rings*, Communications in Algebra, **38: 3** (2010) 988–994.
[3] G. Dolinar, B. Kuzma, P. Oblak, *On maximal distances in a commuting graph*. Electron. J. Linear Algebra **23** (2012) 243–256.
[4] R. M. Solomon and A.J. Woldar, *Simple groups are characterized by their non-commuting graphs*. J. Group Theory **16**, No. 6, (2013) 793–824.

Proof that $d(B, J) = 4$

$$\text{WLOG} \quad B = \begin{pmatrix} 0 & \ldots\ldots\ldots\ldots & & & 0 & -m_0 \\ 1 & 0 & \ldots\ldots\ldots & & 0 & -m_1 \\ 0 & 1 & 0 & \ldots\ldots & 0 & -m_2 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & \ldots\ldots\ldots & & 1 & 0 & -m_{n-2} \\ 0 & 0 & \ldots\ldots & 0 & 1 & -m_{n-1} \end{pmatrix}.$$

Proof that $d(B, J) = 4$

- If $B — X — Y — J$ then $Y \in \mathcal{C}(J) = \mathrm{Poly}(J)$.
  WLOG, $Y^2 = 0$.

Proof that $d(B, J) = 4$

- If $B \text{—} X \text{—} Y \text{—} J$ then $Y \in \mathcal{C}(J) = \mathrm{Poly}(J)$.
  WLOG, $Y^2 = 0$.
- $$\text{HENCE} \quad Y = \begin{pmatrix} \mathbf{0}_{(n-r),(n-r)} & 0 & \widehat{D}_{13} \\ 0 & \mathbf{0}_{(2r-n),(2r-n)} & 0 \\ 0 & 0 & \mathbf{0}_{(n-r),(n-r)} \end{pmatrix}$$

Proof that $d(B, J) = 4$

- If $B \text{---} X \text{---} Y \text{---} J$ then $Y \in \mathcal{C}(J) = \text{Poly}(J)$.
  WLOG, $Y^2 = 0$.
- $$\text{HENCE} \quad Y = \begin{pmatrix} \mathbf{0}_{(n-r),(n-r)} & 0 & \widehat{D}_{13} \\ 0 & \mathbf{0}_{(2r-n),(2r-n)} & 0 \\ 0 & 0 & \mathbf{0}_{(n-r),(n-r)} \end{pmatrix}$$
- Easy to calculate that $X = (x_{ij})_{1 \leq i,j \leq 3}$ commutes with Y iff

$$X = \begin{pmatrix} \star & \star & \star \\ \mathbf{0}_{(2r-n),(n-r)} & \star & \star \\ \mathbf{0}_{(n-r),(n-r)} & \mathbf{0}_{(n-r),(2r-n)} & \star \end{pmatrix}.$$

Proof that $d(B, J) = 4$

$$B - X - Y - J; \qquad X = \begin{pmatrix} \star & \star & \star \\ \mathbf{0}_{(2r-n),(n-r)} & \star & \star \\ \mathbf{0}_{(n-r),(n-r)} & \mathbf{0}_{(n-r),(2r-n)} & \star \end{pmatrix}.$$

- However, X also commutes with B, so $X = \sum_{i=0}^{n-1} \lambda_i B^i$.

Proof that $d(B, J) = 4$

$$B - X - Y - J; \qquad X = \begin{pmatrix} \star & \star & \star \\ \mathbf{0}_{(2r-n),(n-r)} & \star & \star \\ \mathbf{0}_{(n-r),(n-r)} & \mathbf{0}_{(n-r),(2r-n)} & \star \end{pmatrix}.$$

- However, X also commutes with B, so $X = \sum_{i=0}^{n-1} \lambda_i B^i$.
- Considering the images of standard basis vectors,

$$B^i = \begin{pmatrix} \mathbf{0}_{i,(n-i)} & \star_{i,i} \\ \mathrm{Id}_{n-i} & \star_{(n-i),i} \end{pmatrix}; \qquad (i = 0, \ldots, n-1).$$

Proof that $d(B, J) = 4$

$$B - X - Y - J; \qquad X = \begin{pmatrix} \star & \star & \star \\ \mathbf{0}_{(2r-n),(n-r)} & \star & \star \\ \mathbf{0}_{(n-r),(n-r)} & \mathbf{0}_{(n-r),(2r-n)} & \star \end{pmatrix}.$$

- However, X also commutes with B, so $X = \sum_{i=0}^{n-1} \lambda_i B^i$.
- Considering the images of standard basis vectors,

$$B^i = \begin{pmatrix} \mathbf{0}_{i,(n-i)} & \star_{i,i} \\ \mathrm{Id}_{n-i} & \star_{(n-i),i} \end{pmatrix}; \qquad (i = 0, \ldots, n-1).$$

- $(n1)$-entry of X must be zero, so $\lambda_{n-1} = 0$.

Proof that $d(B, J) = 4$

$$B — X — Y — J; \qquad X = \begin{pmatrix} \star & \star & \star \\ \mathbf{0}_{(2r-n),(n-r)} & \star & \star \\ \mathbf{0}_{(n-r),(n-r)} & \mathbf{0}_{(n-r),(2r-n)} & \star \end{pmatrix}.$$

- However, X also commutes with B, so $X = \sum_{i=0}^{n-1} \lambda_i B^i$.
- Considering the images of standard basis vectors,

$$B^i = \begin{pmatrix} \mathbf{0}_{i,(n-i)} & \star_{i,i} \\ \mathrm{Id}_{n-i} & \star_{(n-i),i} \end{pmatrix}; \qquad (i = 0, \ldots, n-1).$$

- $(n1)$-entry of X must be zero, so $\lambda_{n-1} = 0$.
- Inductively backwards: Assume $\lambda_{n-1} = 0 = \lambda_{n-2} = \cdots = \lambda_{n-(k-1)}$.
  THEN, B^k is the only power among the remaining powers of B with k-th subdiagonal nonzero. In fact, this subdiagonal has 1 on its every entry. Since $0 < n - k$, it intersects one of the two zero blocks in X, and $\lambda_{n-k} = 0$. So B scalar matrix.