On the estimation of limit cycles number for some planar autonomous system

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a joint work with

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1. Introduction and motivation

We consider the following class of planar autonomous differential systems depending on a real parameter μ

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \mu \sum_{j=0}^{3} h_j(x, \mu) y^j. \tag{1}$$

We assume the functions $h_j, j = 0, ..., 3$, to be continuous in both variables and continuously differentiable in the first variable, moreover we suppose

$$h_3(x, \mu) \not\equiv 0. \tag{2}$$

For $\mu = 0$, system (1) presents a linear conservative system having the first integral $x^2 + y^2 = c^2 > 0$, where c is any real number. If μ crosses zero, then from some circles $x^2 + y^2 = c_i^2$ limit cycles can bifurcate. *Limit cycle* represents an isolated closed trajectory of system (1).

A famous example is the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad (3)$$

where a unique limit cycle bifurcates from the circle $x^2 + y^2 = 2$ as μ crosses zero.

Concerning this bifurcation problem the question arises: How many limit cycles of system (1) can bifurcate from the continuum of circles surrounding the origin as μ crosses zero.

Here we address some inverse problem: How to construct functions $h_j, j = 0, ..., 3$, such that system (1) has not more than a given number N of limit cycles on the whole phase plane for μ belonging to some (global) interval M which M contains the value 0.

Our approach to treat this problem is based on the construction of suitable Dulac-Cherkas functions.

2. The Dulac-Cherkas (D-C) function and its main properties

We recall the definition of a Dulac function for the planar differential system

$$\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y) \tag{4}$$

in some open region $\mathcal{G} \subset R^2$.

Definition 1

Let $P, Q \in C^1(\mathcal{G}, R)$, let X be the vector field defined by (4). A function $B \in C^1(\mathcal{G}, R)$ is called a Dulac function of (4) in $\mathcal{G}$ if the expression

$$div(BX) \equiv \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \equiv (gradB, X) + B\, divX$$

does not change sign in $\mathcal{G}$ and vanishes only on a set $\mathcal{N}$ of measure zero.

The existence of a Dulac function implies the following estimate of the number of limit cycles of system (4) in $\mathcal{G}$.

Proposition 1

Let $\mathcal{G}$ be a p-connected ($p \geq 1$) region in R^2, let $P, Q \in C^1(\mathcal{G}, R)$. If there is a Dulac function B of (4) in $\mathcal{G}$, then (4) has not more than $p-1$ limit cycles in $\mathcal{G}$.

However, the method itself provides no way for the construction of the function B and for the localization of limit cycles lying in the region $\mathcal{G}$.
The most applied forms of B were $x^a y^b$, $a, b \in R$ and $e^{x^a y^b}$.
The method of Dulac function has been generalized in different ways.
One generalization is due to L. A. Cherkas in 1997. The corresponding generalized Dulac function, which we called Dulac-Cherkas function, is defined as follows.

Definition 2

Let $P, Q \in C^1(\mathcal{G}, R)$. A function $\Psi \in C^1(\mathcal{G}, R)$ is called a Dulac-Cherkas function of system (4) in $\mathcal{G}$ if there exists a real number $k \neq 0$ such that

$$\Phi := (grad\ \Psi, X) + k\Psi\ div\ X > 0 \quad (< 0) \quad in \quad \mathcal{G}. \tag{5}$$

Lemma 1

Let $\Omega \subset D$ be connected, let Ψ be a DC function in $\mathcal{G}$. Then $B := |\Psi|^{1/k}$ is a Dulac function in each subregion of $\mathcal{G}$ where Ψ is positive or negative.

The main properties of D-C function can be described with the help of the subset W of $\mathcal{G}$ defined by

$$\mathcal{W} := \{(x, y) \in \mathcal{G} : \Psi(x, y) = 0\}. \tag{6}$$

Lemma 2

Any trajectory of system (4) meeting the curve W intersects W transversally.

Lemma 3

The curve W does not contain any equilibrium of system (4).

Lemma 4

Let W_1 *and* W_2 *be two different smooth local open branches of the curve* W *such that* $\overline{W_1 \cup W_2}$ *is not connected, that is,* $\partial W_1 \cap \partial W_2$ *is empty. Then* W_1 *and* W_2 *do not meet.*

Lemma 5

The curve W *decomposes the region* $\mathcal{G}$ *in subregions on which* Ψ *is definite and the transition from one subregion to an adjacent subregion is connected with a sign change of* Ψ*.*

Theorem 1

Let Ψ *be a DC function of system* (4) *in* $\mathcal{G}$*. Then any limit cycle of system* (4) *which is entirely located in* $\mathcal{G}$ *does not intersect the curve* W*.*

The following facts can be found in **[Cherkas L.A., 1997]** and **[Grin A.A., Schneider K.R. 2007]**.

Theorem 2

Let Ψ be a Dulac-Cherkas function of (4) in $\mathcal{G}$. Then any limit cycle Γ of (4) in $\mathcal{G}$ is hyperbolic and its stability is determined by the sign of the expression $k\Phi\Psi$ on Γ.

The sign of k plays the essential role.

Theorem 3

Let $\mathcal{G}$ be a p-connected region, let Ψ be a D-C function of (4) for $k < 0$ in $\mathcal{G}$ such that $\mathcal{W}$ has s ovals in $\mathcal{G}$. Then system (4) has at most $p - 1 + s$ limit cycles in $\mathcal{G}$, and all limit cycles are hyperbolic.

Remark 1

Condition (5) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set of measure zero, and that no simply closed curve (oval) of this set is a limit cycle of (4).

This approach was exploited also by

Gasull A., Giacomini H. *A new criterion for controlling the number of limit cycles of some generalized Liénard equations (2002)*,

Gasull A., Giacomini H. *Upper bounds for the number of limit cycles through linear differential equations (2006)*,

Gasull A., Giacomini H., Llibre J. *New criteria for the existence and non-existence of limit cycles in Liénard differential systems (2008)*,

Gasull A., Giacomini H. *Upper bounds for the number of limit cycles of some planar polynomial differential systems (2008)*.

and other papers.

3. General idea for the application of DC function to some classes of system (4)

For strip region $\mathcal{G} = \Omega_x = \{(x, y) : x \in [x_1, x_{N_0}], y \in R\}$ we construct the function Ψ in the form

$$\Psi(x, y) = \sum_{j=0}^{n} \Psi_j(x) y^j, \quad \Psi_j \in C^1(R), \tag{7}$$

for systems

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \sum_{j=0}^{l} h_j(x) y^j, \quad h_j \in C^0(R),$$

with $l \geq 1$.

$$\Phi(x, y) \equiv \sum_{i=0}^{m} \Phi_i(x) y^i, \tag{8}$$

where $\Phi_i(x)$ are functions of the known coefficient functions $h_0(x), ..., h_l(x)$, of the unknown coefficient functions $\Psi_0(x), ..., \Psi_n(x)$, of their first derivatives $\Psi_0'(x), ..., \Psi_n'(x)$, and of k.
The highest power m of y in (20) is $m = max\{n+1, n+l-1\}$.
To determine the functions $\Psi_j(x), j = 0, ..., n$, and the real number k we reduce $\Phi(x, y)$ to the following form

$$\Phi(x, y) = \Phi_0(x),$$

satisfying relations

$$\Phi_i(x) \equiv 0 \quad \text{for} \quad i = 1, ..., m. \tag{9}$$

For $l = 1$ and $l = 2$ the relations (9) represent a system of $n + 1$ linear differential equations to determine the $n + 1$ functions $\Psi_j, j = 0, ..., n$. In case $l = 1$ we have Liènard system

$$\dot{x} = y,\ \dot{y} = h_0(x) + h_1(x)y, \tag{10}$$

and the system (9) can be solved successively by simple quadratures, starting with Ψ_n.

$$\begin{aligned}
0 &\equiv \Psi_n'(x), \\
0 &= \Psi_{n-1}'(x) + (k+n)h_1(x)\Psi_n(x), \\
0 &\equiv \Psi_{n-2}'(x) + (k+n-1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
0 &\equiv \Psi_1'(x) + (k+2)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x), \\
0 &\equiv \Psi_0'(x) + (k+1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).
\end{aligned} \tag{11}$$

The general solution depends on $n + 1$ integration constants and on the constant k.

In case $l = 2$ the system (9) can also be integrated by solving inhomogeneous linear differential equations, starting with Ψ_n.

$$
\begin{aligned}
0 &\equiv \Psi_n'(x) + (2k+n)h_2(x)\Psi_n(x), \\
0 &\equiv \Psi_{n-1}'(x) + (2k+n-1)h_2(x)\Psi_{n-1}(x) \\
&\quad + (k+n)h_1(x)\Psi_n(x), \\
0 &\equiv \Psi_{n-2}'(x) + (2k+n-2)h_2(x)\Psi_{n-2}(x) \\
&\quad + (k+n-1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &\equiv \Psi_1'(x) + (2k+1)h_2(x)\Psi_1(x) \\
&\quad + (k+2)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x), \\
0 &= \Psi_0'(x) + 2kh_2(x)\Psi_0(x) \\
&\quad + (k+1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).
\end{aligned}
\tag{12}
$$

The functions Ψ_j depend on the parameter k, but we get no restriction on k in the process of solving this system. To fulfill the condition (5) we have to choose k and the integration constants appropriately.

In case $l = 3$ (Kukles system) the first equation of the system (9) is an algebraic equation which determines the constant k uniquely as $k = -\frac{n}{3}$. The remaining equations represent a system of $n + 1$ linear differential equations. Its general solution depends on $n + 1$ integration constants which can be used to try to fulfill the relations (5).

$$
\begin{aligned}
0 &\equiv (n + 3k)h_3(x)\Psi_n(x), \\
0 &\equiv \Psi_n'(x) + (2k + n)h_2(x)\Psi_n(x) \\
&\quad + (n - 1 + 3k)h_3(x)\Psi_{n-1}(x), \\
0 &\equiv \Psi_{n-1}'(x) + (n - 1 + 2k)h_2(x)\Psi_{n-1}(x) \\
&\quad + (n + k)h_1(x)\Psi_n(x) + (n - 2 + 3k)h_3(x)\Psi_{n-2}, \\
0 &\equiv \Psi_{n-2}'(x) + (2k + n - 2)h_2(x)\Psi_{n-2}(x) \\
&\quad + (k + n - 1)h_1(x)\Psi_{n-1}(x) + nh_0(x)\Psi_n(x) \qquad (13) \\
&\quad + (n - 3 + 3k)h_3(x)\Psi_{n-3}(x), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
0 &\equiv \Psi_1'(x) + (1 + 2k)h_2(x)\Psi_1(x) + 3kh_3(x)\Psi_0(x) \\
&\quad + (2 + k)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x), \\
0 &\equiv \Psi_0'(x) + 2kh_2(x)\Psi_0(x) \\
&\quad + (k + 1)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x).
\end{aligned}
$$

In the case of system (1) ($l = 3,\ n = 2$)

$$\begin{aligned}
\Phi_4(x,\mu) &\equiv\ (2+3k)\mu h_3(x,\mu)\Psi_2(x,\mu),\\
\Phi_3(x,\mu) &\equiv\ \Psi_2'(x,\mu)\\
&+ (2k+2)\mu h_2(x,\mu)\Psi_2(x,\mu) + (1+3k)\mu h_3(x,\mu)\Psi_1(x,\mu),\\
\Phi_2(x,\mu) &\equiv\ \Psi_1'(x,\mu) + (1+2k)\mu h_2(x,\mu)\Psi_1(x,\mu)\\
&+ (2+k)\mu h_1(x,\mu)\Psi_2(x,\mu) + 3k\mu h_3(x,\mu)\Psi_0(x,\mu),\\
\Phi_1(x,\mu) &\equiv\ \Psi_0'(x,\mu) + 2k\mu h_2(x,\mu)\Psi_0(x,\mu)\\
&+ (k+1)\mu h_1(x,\mu)\Psi_1(x,\mu) + 2\mu h_0(x,\mu)\Psi_2(x,\mu) - 2x\Psi_2(x,\mu).
\end{aligned} \tag{14}$$

In all cases of l and n

$$\Phi_0(x,\mu) \equiv -\Psi_1(x,\mu)x + \mu\Big(k\Psi_0(x,\mu)h_1(x,\mu) + \Psi_1(x,\mu)h_0(x,\mu)\Big). \tag{15}$$

For such purpose we can apply the reduction to the linear programming problem

$$L \to \max, \ \sum_{i=0}^{n} C_i \tilde{\Phi}_i(x_l) - L \geq 0, \ |C| \leq 1, \tag{16}$$

$x_l \in [x_1, x_{N_0}], \ i = \overline{1, N_0}$.
If this is not possible we can reduce function $\Phi(x, y)$ to one of the following forms

$$\Phi(x, y) = \Phi_0(x) + \Phi_1(x)y + \Phi_2(x)y^2,$$

$$\Phi(x, y) = \Phi_0(x) + \Phi_2(x)y^2 + \Phi_4(x)y^4$$

The paper **[Cherkas L.A., Grin A., 2010]** contains two algorithms to construct $\Phi(x, y) > 0$: for odd y^p all $\Phi_p(x) = 0$, for even y^p all $\Phi_p(x) \geq 0$ and $\Phi_0(x) > 0$.
Or we have to look for corresponding conditions on the functions h_i.

For the system

$$\dot{x} = yP_0(x), \ \dot{y} = h_0(x) + h_1(x)y + h_2(x)y^2 + h_3y^3, \qquad (17)$$

with $P_0(x) \in C^1$, to fulfil $\Phi(x,y) > 0$ we require $|\Phi_w(x)| < \varepsilon$ for odd y^w, ε sufficiently small and $\Phi_v(x) \geq 0$ for even y^v and $\Phi_0(x) > 0$. In this case we take all $\Psi_i(x)$ in the form $\Psi_i(x) = \sum_{j=0}^{m_i} C_{ij}x^j$, $C_{ij} \in \mathbb{R}$, $m_i \in N$ and solve the linear programming problem

$$L \to max, \ \sum_{j=0}^{m} C_j \Phi_{vj}(x_l) - L > 0, \ |\sum_{j=0}^{m} C_j \Phi_{wj}(x_l)| - \varepsilon < 0,$$

the vector C_j consists of coefficients C_{ij} from all $\Psi_i(x)$ and has dimension $m = m_1 + \ldots + m_n + n$.

Example 1

For system (17) with $P_0(x) = 1 + x^2/6$, $h_0(x) = -x(1+x^2)$, $h_1(x) = 1 - x^2$, $h_2(x) = (x-1)/100$, $h_3 = -1$ function Ψ is constructed by using $k = -1$, $n = 2$, $m_1 = 6$, $m_2 = 7$, $m_3 = 8$, $\varepsilon = 0.0000001$, $N_0 = 100$, $[-1.5; 1.5]$. For the solution (C^*, L^*) the equation $\Psi = 0$ defines unique oval and polynomial $\Phi(x,y) > 0$ on the whole plane. It allows to prove the uniqueness of limit cycle globally.

In case $l \geq 4$ system (9) consist of $n+1$ linear differential equations and $l-2$ algebraic equations to determine k and the functions $\Psi_0, ..., \Psi_n$. Thus, this system has generically no solution.
In **[Cherkas L.A., Grin A., Schneider K.R. 2011]** it was shown that under additional conditions on the functions h_i system (9) has a nontrivial solution which satisfies the inequalities (5).
In **[X. Ioakim 2014]** it is proved the uniqueness of the limit cycle on the whole phase plane for generalized Van der Pol system

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon y^{2m+1}(1 - x^{2q}),$$

where ε is a small parameter tending to zero, m and $q \in N$.
To prove this result for global interval of ε we constructed $\Psi = x^2 + y^2 - 1$. The corresponding function $\Phi = 2\varepsilon y^{2m}(x^2 - 1)^2(1 + x^2 + x^4 + ... + x^{2q-2})$.
In the same manner for the system

$$\dot{x} = y^{2m-1}, \quad \dot{y} = -x^{2q-1} + \varepsilon y^{2m+1}(1 - x^{2q}),$$

where unperturbed system has the Hamiltonian $x^{2q}/2q + y^{2m}/2m = c$ we constructed $\Psi = m/q(x^{2q} + q/py^{2m} - 1)$. The corresponding function $\Phi = y^{2m}\varepsilon 2m^2 c_2/q(x^{2q} - 1)^2$.

For the sequel we suppose $\mathcal{G}$ to be a simply connected region containing the origin and assume that the Dulac-Cherkas function Ψ is a polynomial in y

$$\Psi(x, y, \mu) = \sum_{j=0}^{n} \Psi_j(x, \mu) y^j \tag{18}$$

with

$$\Psi_n(x, \mu) \not\equiv 0. \tag{19}$$

Then, the corresponding function Φ is in case of system (1)

$$\Phi(x, y, \mu) = \sum_{i=0}^{m} \Phi_i(x, \mu) y^i, \quad m = n + 2. \tag{20}$$

We consider the cases $n = 1$ and $n = 2$. Thus, system (1) has no limit cycle in case $n = 1$ and at most one limit cycle in case $n = 2$ in $\mathcal{G}$.

4. Construction of systems with no limit cycle

In the case $n = 1$ we have the representations

$$\Psi(x, y, \mu) = \Psi_0(x, \mu) + \Psi_1(x, \mu)y \tag{21}$$

with

$$\Psi_1(x, \mu) \not\equiv 0. \tag{22}$$

$$\Phi(x, y, \mu) = \sum_{i=0}^{3} \Phi_i(x, \mu)y^i. \tag{23}$$

where

$$\Phi_0(x, \mu) \equiv -\Psi_1(x, \mu)x + \mu\Big(k\Psi_0(x, \mu)h_1(x, \mu) + \Psi_1(x, \mu)h_0(x, \mu)\Big). \tag{24}$$

The relation for the function Φ_0 is valid for any n.
To derive conditions on the coefficient functions h_j such that one of the inequalities in (5) is fulfilled we study the following cases $\Phi(x, y, \mu) \equiv \Phi_0(x, \mu)$ and $\Phi(x, y, \mu) \equiv \Phi_0(x, \mu) + \Phi_2(x, \mu)y^2$.

4.1. Nonexistence of limit cycles if Φ does not depend on y

If we set

$$\Psi_0(x,\mu) := q \neq 0, \quad \Psi_1(x,\mu) := \mu x \tag{25}$$

we derive conditions on k and the functions h_j.

$$k = -\frac{1}{3}. \tag{26}$$

$$h_3(x,\mu) := \frac{3q + \mu^2 x^2 h_1(x,\mu)}{3q^2}. \tag{27}$$

$$h_2(x,\mu) := \frac{\mu x h_1(x,\mu)}{q}. \tag{28}$$

Taking into account

$$\Phi(x,y,\mu) \equiv \Phi_0(x,\mu) \equiv -\mu\big(x^2 + \frac{q}{3}h_1(x,\mu) - \mu x h_0(x,\mu)\big). \qquad (29)$$

and that system (1) has no limit cycle for $\mu = 0$, we have the result:

Theorem 4

Let q be any given real number different from zero, let $h_0, h_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions, let h_2 and h_3 be defined by (28) and (27), respectively. If there exists an interval M such that for $\mu \in M$ the expression

$$-x^2 - \frac{q}{3}h_1(x,\mu) + \mu x h_0(x,\mu)$$

has the same sign for all $x \in \mathbb{R}$ and does not vanish identically for any x-interval, then system (1) has no limit cycle for $\mu \in M$.

As an example we consider the case

$$q = -3, \quad h_1(x, \mu) \equiv x^2 \tag{30}$$

and obtain $\Phi(x, y, \mu) \equiv \mu^2 x h_0(x, \mu)$ and $\Psi(x, \mu) \equiv q + \mu xy$. Thus, we have:

Corollary 1

The autonomous system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + \mu\Big(h_0(x, \mu) + x^2 y - \frac{\mu}{3}x^3 y^2 + \frac{-9 + \mu^2 x^4}{27} y^3\Big) \end{aligned}$$

has no limit cycle for any μ provided that for any $\mu \neq 0$ the function $xh_0(x, \mu)$ does not change sign for $x \in \mathbb{R}$ and does vanish identically for any x-interval.

The way we used to derive conditions for system (1) to have no limit cycle can be characterized as an algebraic method: we prescribe Ψ_0 and Ψ_1 and determine conditions for the coefficient functions h_j, $0 \leq j \leq 3$, by solving the identities for $\Phi_3(x,\mu)$, $\Phi_2(x,\mu)$, $\Phi_1(x,\mu)$ in (9) and the inequality $\Phi_0(x,\mu) > 0\,(< 0)$

Now we describe another so called algebraic-differential approach based on a combination of the approach used above and the method used in **[Cherkas L.A., Grin A.A., Shcneider K.R. 2011]**. As in the preceding approach we first determine the number k in order to satisfy the identity $\Phi_3(x,\mu) \equiv 0$. Then we solve the identities $\Phi_2(x,\mu) \equiv 0$ and $\Phi_1(x,\mu) \equiv 0$ as a system of non-homogeneous linear differential equations for Ψ_0 and Ψ_1. In general it is not possible to get an explicit solution of this system. Under the assumption that we are able to obtain a solution of that system as a function of the coefficient functions h_j, we can plug in this solution into the inequality (5). By this way we derive conditions on the coefficient functions h_j implying that Ψ is a Dulac-Cherkas function.

As an example we consider system (1) under the condition

$$h_2(x,\mu) \equiv 0. \tag{31}$$

From the first identity in (9) we get $k = -1/3$, the identities for Φ_2 and Φ_1 read

$$\begin{aligned} \Phi_2(x,\mu) &\equiv \Psi_1'(x,\mu) - \mu h_3(x,\mu)\Psi_0(x,\mu) \equiv 0, \\ \Phi_1(x,\mu) &\equiv \Psi_0'(x,\mu) + \frac{2}{3}\mu h_1(x,\mu)\Psi_1(x,\mu) \equiv 0. \end{aligned} \tag{32}$$

We consider (32) as a system of linear homogeneous differential equations to determine Ψ_0 and Ψ_1. If we look for a solution of system (32) satisfying

$$\Psi_1(x,\mu) \equiv \kappa\Psi_0(x,\mu), \tag{33}$$

where κ is some constant which can depend on the parameter μ, we obtain the condition

$$h_3(x,\mu) \equiv -\frac{2}{3}\kappa^2 h_1(x,\mu). \tag{34}$$

Therefore, we get from the last differential equation in (32) the special solution

$$\Psi_0(x,\mu) \equiv \exp\Big(-\frac{2}{3}\mu\kappa\int^x h_1(\xi,\mu)d\xi\Big). \tag{35}$$

Finally, we obtain

$$\Phi_0(x,\mu) = -\frac{\mu^2}{3}\exp\Big(\frac{\mu^2}{9}x^2\Big)h_0(x,\mu), \quad \Psi(x,y,\mu) = \exp\Big(\frac{\mu^2}{9}x^2\Big)\Big(1-\frac{\mu}{3}y\Big).$$

Thus, we have the result:

Theorem 5

Let $h_0 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous function and for any μ does not change sign and does not vanish identically in x on any x-interval, then the autonomous system

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -x+\mu\Big(h_0(x,\mu)+xy-\frac{2}{27}\mu^2xy^3\Big),\end{aligned} \tag{36}$$

has no limit cycle in the phase plane for any μ.

4.2. Nonexistence of limit cycles if Φ_3 and Φ_1 vanish identically

In what follows we have

$$\Phi(x, y, \mu) = \Phi_0(x, \mu) + \Phi_2(x, \mu) y^2. \tag{37}$$

As in the subsection before, we suppose $\Psi(x, y, \mu) \equiv q + \mu xy$. Solving the identities $\Phi_3 \equiv 0$ and $\Phi_1 \equiv 0$ we get

$$\Phi_2(x, \mu) \equiv \mu\Big(1 - q h_3(x, \mu) + \frac{\mu^2}{3q} x^2 h_1(x, \mu)\Big), \tag{38}$$

$$\Phi_0(x, \mu) \equiv \mu\Big(-x^2 - \frac{q}{3} h_1(x, \mu) + \mu x h_0(x, \mu)\Big). \tag{39}$$

The relation

$$\Phi_2(x,\mu)\Phi_0(x,\mu) \geq 0, \tag{40}$$

is a sufficient condition for Φ to have the same sign. Using (38) and (39) it reads

$$\mu^2\Big(-x^2 - \frac{q}{3}h_1(x,\mu) + \mu x h_0(x,\mu)\Big) \times \Big(1 - qh_3(x,\mu) + \frac{\mu^2}{3q}x^2 h_1(x,\mu)\Big) \geq 0. \tag{41}$$

Theorem 6

Let q be any given real number different from zero, let $h_0, h_1, h_3 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous functions, let the function h_2 be defined by (28). Suppose the existence of an interval M such that for $\mu \in M$

(i). Φ_0 and Φ_2 do not vanish identically zero at the same time for any x-interval.

(ii). The inequality (41) is valid for all $x \in \mathbb{R}$.

Then system (1) has no limit cycle for $\mu \in M$.

In the special case $q = -3$ and $h_1(x, \mu) \equiv x^2$ we have the result

Corollary 2

Let $h_0, h_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying for $\mu \in M$ and $x \in \mathbb{R}$

$$\mu x h_0(x, \mu)\Big(1 + 3h_3(x, \mu) - \frac{\mu^2}{9}x^4\Big) \geq 0$$

then the autonomous system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + \mu\Big(h_0(x, \mu) + x^2 y - \frac{\mu x^3}{3}y^2 + h_3(x, \mu)y^3\Big). \end{aligned} \tag{42}$$

has no limit cycle in the phase plane for any μ.

5. Construction of systems having at most one limit cycle

In this section we consider the case $n = 2$

$$\Psi(x, y, \mu) = \Psi_0(x, \mu) + \Psi_1(x, \mu)y + \Psi_2(x, \mu)y^2, \tag{43}$$

$$\Phi(x, y, \mu) = \sum_{i=0}^{4} \Phi_i(x, \mu)y^i. \tag{44}$$

The case $n = 2$ implies that the set $\mathcal{W}_\mu$ consists of at most one oval.
To derive conditions on the functions h_j we study in the following subsections the cases $\Phi(x, y, \mu) \equiv \Phi_0(x, \mu)$,
$\Phi(x, y, \mu) \equiv \Phi_0(x, \mu) + \Phi_2(x, \mu)y^2$,
$\Phi(x, y, \mu) \equiv \Phi_0(x, \mu) + \Phi_2(x, \mu)y^2 + \Phi_4(x, \mu)y^4$.
In all cases we apply the algebraic approach, that is, we prescribe the function $\Psi(x, y, \mu)$.

5.1. Existence of at most one limit cycle if Φ does not depend on y

Concerning Ψ we assume

$$\Psi(x, y, \mu) \equiv px^2 - c + \mu xy + py^2. \tag{45}$$

Thus, under the conditions

$$p > 0, \quad 4p^2 - \mu^2 > 0, \quad c > 0 \tag{46}$$

the set $\mathcal{W}_\mu$ consists exactly of one oval which is an ellipse.

We get

$$k = -\frac{2}{3}. \tag{47}$$

$$h_2(x,\mu) := \frac{3}{2p}\mu x h_3(x,\mu). \tag{48}$$

$$h_1(x,\mu) := \frac{3}{8p^2}\Big(4ph_3(x,\mu)(px^2-c) + h_3(x,\mu)\mu^2x^2 - 2p\Big). \tag{49}$$

$$h_0(x,\mu) := \frac{\mu}{16p^3}\Big(12ph_3(x,\mu)x(px^2-c) - \mu^2h_3(x,\mu)x^3 + 2px\Big). \tag{50}$$

$$\Phi_0(x,\mu) \equiv \frac{\mu}{16p^3}\Big(-x^4h_3(x,\mu)(4p^2-\mu^2)^2 - x^2 2p(1-4ch_3(x,\mu))(4p^2-\mu^2) -$$

$$- 8\mathrm{p}^2c(1+2ch_3(x,\mu))\Big)$$

A detailed analysis of $\Phi_0(x, \mu)$ provides the result

Lemma 6

Suppose the following conditions are satisfied:
(A_1). *Let* c *and* p *be given positive numbers, let* μ *be a number of the interval* $(-2p, 2p)$.
(A_2). *Let* $h_3 : \mathbb{R} \times (-2p, 2p) \to \mathbb{R}$ *be a continuous function satisfying*

$$h_3(x, \mu) > \frac{1}{16c} \quad \text{for} \quad (x, \mu) \in \mathbb{R} \times (-2p, 2p). \tag{51}$$

Then the function $\Phi_0(x, \mu)$ *is negative (positive) definite for* $(x, \mu) \in \mathbb{R} \times (0, 2p)$ $\Big((x, \mu) \in \mathbb{R} \times (-2p, 0)\Big)$.

Additionally to the assumptions (A_1) and (A_2) we suppose
(A_3). For $j = 0, 1, 2$, the functions $h_j : \mathbb{R} \times (-2p, 2p) \to \mathbb{R}$ are defined by (50), (49) and (48), respectively.

Theorem 7

Under the assumptions $(A_1) - (A_3)$ *system (1) has at most one limit cycle in the phase plane. If system (1) has a limit cycle* Γ_μ, *then it is hyperbolic and contains the ellipse* $\mathcal{W}_\mu$ *in its interior.*

5.2. Existence of at most one limit cycle if Φ_4, Φ_3 and Φ_1 vanish identically

Concerning the function Ψ we assume to have the form

$$\Psi(x, y, \mu) = px^2 + py^2 - c, \tag{52}$$

where p and c are positive numbers.
By using this approach we get

$$\Phi_2(x, \mu) = \mu\Big(\frac{4}{3}h_1(x, \mu)p - 2h_3(x, \mu)(px^2 - c)\Big) \tag{53}$$

$$\Phi_0(x, \mu) = \mu\Big(-\frac{2}{3}(px^2 - c)h_1(x, \mu)\Big). \tag{54}$$

Putting

$$h_1(x,\mu) := px^2 - c, \quad h_3(x,\mu) := px^2 - c + \frac{2}{3}p \tag{55}$$

we obtain

$$\Phi_2(x,\mu) = -2\mu(px^2 - c)^2, \quad \Phi_0(x,\mu) = -\frac{2}{3}\mu(px^2 - c)^2 \tag{56}$$

Therefore, the condition $\Phi_2(x,\mu)\Phi_0(x,\mu) \geq 0$ holds and we have the result:

Theorem 8

The autonomous system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + \mu\Big((px^2 - c)y + (px^2 - c + \frac{2}{3}p)y^3\Big) \end{aligned} \tag{57}$$

has for any positive numbers p and c at most one limit cycle in the whole phase plane.

5.3. Existence of at most one limit cycles if Φ_3 and Φ_1 vanish identically

In this case the function $\Phi(x, y, \mu)$ has the form
$\Phi(x, y, \mu) \equiv \Phi_0(x, \mu) + \Phi_2(x, \mu)y^2 + \Phi_4(x, \mu)y^4$. Hence, one from the following conditions

$$\text{functions} \quad \Phi_0(x, \mu), \Phi_2(x, \mu), \Phi_4(x, \mu) \quad \text{have the same sign} \tag{58}$$

$$D := \Phi_2^2(x, \mu) - 4\Phi_0(x, \mu)\Phi_4(x, \mu) \leq 0 \tag{59}$$

implies that $\Phi_(x, y.\mu)$ does not change sign.
As $\Psi(x, y, \mu)$ we choose the function $\Psi(x, y, \mu) = x^2 + y^2 - 1$

Putting

$$k = -1, \tag{60}$$

$$h_0(x,\mu) := h_2(x,\mu)(x^2-1) \tag{61}$$

we obtain

$$\Phi_4(x,\mu) = -\mu h_3(x,\mu), \quad \Phi_2(x,\mu) = \mu h_1(x,\mu) - 3\mu h_3(x,\mu)(x^2-1),$$

$$\Phi_0(x,\mu) = -\mu h_1(x,\mu)(x^2-1).$$

And the inequality (59) reads

$$\mu^2(h_1(x,\mu) - 3h_3(x,\mu)(x^2-1))^2 - 4\mu^2 h_3(x,\mu)\mu h_1(x,\mu)(x^2-1) \leq 0. \tag{62}$$

Therefore, we have the result:

Theorem 9

Let $h_1, h_2, h_3 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous functions, let the function h_0 be defined by (61). Additionally we assume that the functions h_1 and h_3 are such that inequality (62) is valid for $(x,\mu) \in \mathbb{R}\times\mathbb{R}$ or functions$\Phi_0(x,\mu), \Phi_2(x,\mu), \Phi_4(x,\mu)$ have the same sign. Then the system (1) has at most one limit cycle.

To derive $\Phi_0(x, y, \mu)$ which has the same sign for all $x \in \mathbb{R}$ we choose

$$h_1(x, \mu) := x^2 - 1. \tag{63}$$

If we additionally suppose

$$h_3(x, \mu) := \frac{x^2}{3}, then \tag{64}$$

$$\Phi(x, y, \mu) = -\mu\Big(\frac{x^2}{3}y^4 + (x^2 - 1)^2 y^2 + (x^2 - 1)^2\Big) > 0(< 0) \tag{65}$$

for $\mu < 0(\mu > 0)$ and $\Phi(x, y, \mu)$ vanishes only on set measure zero.

Corollary 3

Let there exist continuous function $h_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then autonomous system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + \mu\Big((x^2 - 1)h_2(x, \mu) + (x^2 - 1)y + h_2(x, \mu)y^2 + \frac{x^2}{3}y^3\Big) \end{aligned} \tag{66}$$

has at most one limit cycle in the whole phase plane for all $\mu \neq 0$.

6. Conditions for the existence of a unique limit cycle

To be able to formulate the corresponding result introduce the following condition:
(A). The functions $h_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $0 \le j \le 3$, can be represented in the form

$$h_j(x, \mu) = h_j(x, 0) + \tilde{h}_j(x, \mu)\mu,$$

where $\tilde{h}_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.
Under this assumption, system (1) can be written in the following form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \mu q(x, y) + \mu^2 h(x, y, \mu), \tag{67}$$

where

$$q(x, y) := \sum_{j=0}^{3} h_j(x, 0) y^j, \quad h(x, y, \mu) := \sum_{j=0}^{3} \tilde{h}_j(x, \mu) y^j.$$

The application of a well-known theorem **[Andronov A.A. 1973, Theorem 75]** implies the result:

Theorem 10

Suppose the assumption (A) to be valid. If in polar coordinates the equation

$$\int_0^{2\pi} q(r\cos\varphi, r\sin\varphi)\sin\varphi\, d\varphi = 0 \tag{68}$$

has a positive root $r = r_*$ *satisfying*

$$\int_0^{2\pi} \frac{\partial q(r_*\cos\varphi, r_*\sin\varphi)}{\partial y}\, d\varphi \neq 0, \tag{69}$$

then system (67) has for sufficiently small μ *a unique limit cycle near the circle centered at the origin with radius* r_* *which is hyperbolic.*

6.1. Existence of a unique limit cycle in the class of systems considered in subsection 5.1

In section 5.1 we considered systems (1), where the functions h_0, h_1, h_2 are defined by means of the function h_3.
In the special case $c = \frac{1}{4}, p = 1, h_3(x, \mu) \equiv 1$ we have the result

Theorem 11

System (1) with

$$h_3(x, \mu) \equiv 1, \quad h_2(x, \mu) \equiv \frac{3}{2}\mu x,$$

$$h_1(x, \mu) \equiv \frac{3}{8}[(4 + \mu^2)x^2 - 3], \quad h_0(x, \mu) \equiv \frac{\mu x}{16}[12x^2 - 1 - \mu^2 x^2]$$

has for sufficiently small $|\mu| \neq 0$ a unique limit cycle Γ_μ which tends to the unit circle as μ tends to zero.

6.2. Existence of a unique limit cycle in the class of systems considered in subsection 5.2

In the same way we prove the uniqueness of limit cycle for system (57):
$q(x,y) := (px^2 - c)y + (px^2 - c + \frac{2}{3}p)y^3$,

$$\int_0^{2\pi} \left(pr^3 \cos^2\varphi \sin\varphi - cr\sin\varphi + \left(\frac{2}{3}p - c\right) r^3 \sin^3\varphi + pr^5 \cos^2\varphi \sin^3\varphi \right) \sin\varphi \, d$$

$$= r\pi\left(\frac{p}{8}r^4 + \frac{3}{4}(p-c)r^2 - c\right) = 0 \tag{70}$$

$$\int_0^{2\pi} \left(pr_*^2 \cos^2\varphi - c + (3pr_*^2\cos^2\varphi + 2p - 3c) r_*^2 \sin^2\varphi \right) d\varphi \neq 0. \tag{71}$$

The equation (70) has the unique positive solution $r_* = \sqrt{\frac{3(c-p)+4\sqrt{D}}{p}}$, where $D = \frac{9(p-c)^2+8pc}{16}$, which fulfills the inequality (71).

Theorem 12

System (57) under the condition (46) has for sufficiently small $|\mu| \neq 0$ a unique limit cycle Γ_μ which tends to the circle with radius r_* as μ tends to zero.

In the special case $c = 1, p = 1$ we get $r_* = 2/\sqrt[4]{2} \approx 1.68179$.

6.3. Existence of a unique limit cycle in the class of systems considered in subsection 5.3

For system (66)
$q(x, y) := (x^2 - 1)h_2(\varphi, \mu) + (x^2 - 1)y + h_2(\varphi, \mu)y^2 + \frac{1}{2}py^3$, in the case of an even in x function $h_2(\varphi, \mu) := h_2$

$$\int_0^{2\pi} \left(h_2 r^2 \cos^2\varphi - h_2 + r^3 \cos^2\varphi \sin\varphi - r \sin\varphi + h_2 r^2 \sin^2\varphi + \frac{1}{2} r^3 \sin^3\varphi \right) \sin$$

$$= r\pi \left(\frac{5}{8} r^2 - 1 \right) = 0 \tag{72}$$

$$\int_0^{2\pi} \left(r_*^2 \cos^2\varphi - 1 + 2h_2 r_* \sin\varphi + \frac{3}{2} r_*^2 \sin^2\varphi \right) d\varphi \neq 0. \tag{73}$$

The (72) has the unique positive solution $r_* = \sqrt{\frac{8}{5}}$ satisfying (73).

Theorem 13

System (66) for all even in x functions h_2 has for sufficiently small $|\mu| \neq 0$ a unique limit cycle Γ_μ which tends to the circle with radius r_* as μ tends to zero.

Conclusions and possible further development

1. Application to systems with cylindrical phase space;
2. Application to systems in the following form

$$\frac{dx}{dt} = y + \mu \sum_{j=0}^{l-1} d_j(x,\mu) y^j, \quad \frac{dy}{dt} = -x + \mu \sum_{j=0}^{l} h_j(x,\mu) y^j;$$

3. Application to systems where unperturbed system has nonlinear center.

Thank you for your attention!