Hidden Lagrangian constraints and differential Thomas decomposition

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Plan

Constrained dynamics

Local symmetry

All fundamental models in physics (QED, QCD, YM, SM, GR, SUSY, ST,...) are invariant under some local symmetry transformations: gauge (QED, QCD, YM, SM); local supersymmetry (SUSY); space-time diffeomorphisms (GR,ST).

- √ Such models are called constrained models or singular models.
- √ Local symmetry relates different solutions stemming from the same IC (position and velocity).
- √ General solution contains arbitrary time-dependent functions.
- √ A continuous set of accelerations belongs to the same IC.
- √ All accelerations correspond to a subset of IC defined by (hidden) Lagrangian constraints.
- √ **Q.: How to compute them?**

Euler-Lagrange equations

All fundamental laws are understood in terms of action and Hamilton's principle.

Physics (field theories):

$$S = \int dt \int d^3x \mathcal{L}(\varphi^a, \partial_{x_i}\varphi^a, \dot{\varphi}^a) \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} - \frac{\partial \mathcal{L}}{\partial \varphi^a} = 0, \quad \partial_0 \varphi^a \equiv \dot{\varphi}^a$$

Mechanics: (dynamical systems)

$$S = \int dt\, L(q^a, \dot{q}^a) \Longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = 0,$$

Lagrangian density $\mathcal{L}$ and Lagrangian L are (typically) differential polynomials.

Singular models

Lagrangian is (singular) regular if Hessian $H_{i,j}$ is (not)invertible

$$H_{i,j} = \begin{cases} \dfrac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} & \text{(Dynamical System)} \\ \dfrac{\partial^2 \mathcal{L}}{\partial \dot{\varphi}^i \, \partial \dot{\varphi}^j} & \text{(Field Theoretic Model)} \end{cases}$$

In terms of Hessian the set E of Euler-Lagrange equations reads

$$E := \{ \, e_i = 0 \mid i = 1, \ldots, m \, \} \, ,$$

$$e_i := \begin{cases} H_{i,j} \, \ddot{q}^j + P_i & \text{(Dynamical System)} \\ H_{i,j} \, \ddot{\varphi}^j + P_i & \text{(Field Theoretic Model)} \end{cases}$$

$H_{i,j}$ and P_i are differential polynomials of order ≤ 1.

Standard computation via linear algebra (Wipf'1994)

Step 1. Compute Hessian H, derive the set E of Euler-Lagrange equations of cardinality $m := |E|$ and put $C := \{\,\}$.

Step 2. Compute the rank r of the Hessian taking into account equations in E.

Step 3. If $r = m$, then go to Step 6. Otherwise, go to the next step.

Step 4. Compute a basis V of the nullspace of H, set up

$$C := \{\, P_i V^i_\alpha \mid \alpha = 1, \ldots, |V| \,\}$$

and enlarge the equation set

$$E := E \cup \{\, c = 0 \mid c \in C \setminus \{0\} \,\}.$$

Step 5. Set $m := r$ and go to Step 2.

Step 6. Return C.

Pros and cons

- Pros
  - √ Application of computationally efficient linear algebra based methods to test singularity and to construct constraints.
  - √ Linear independence of constraints.
- Cons
  - √ The approach is not completely algorithmic. In particular,
    - It fails to account for the dependence of Hessian rank on area in the space $(\varphi, \partial\varphi)$ or $(q, \dot{q})$.
    - Algebraic completion of constraints needs reduction modulo radical ideal they generate that is very expensive computationally.
    - The output set C of constraints has to be further processed to extract the set of algebraically independent Lagrangian constraints.
  - √ Full or even partial implementation is unknown.

Integrability conditions and involution

Definition

Given a system S of PDEs of order q, its differential consequence of order $\leq q$ is called integrability condition to S.

All integrability conditions are detected and incorporated into the differential system by its completion to involution (Seiler'10).

In general, a nonlinear differential system does not admit its algorithmic completion to involution. Instead, one can decompose it (Thomas decomposition) fully algorithmically into finitely many involutive subsystems with disjoint set of solutions (Bächler, Gerdt, Lange-Hegermann, Robertz'12).

For a linear input system the algorithm performs its completion to involution without splitting.

Ranking of partial derivatives

The output of a Thomas decomposition algorithm is determined by an input differential system and by a ranking of partial derivatives

Definition

A total ordering $\prec$ on the set of partial derivatives is a ranking if for all indices a, b, μ, ν, ρ and multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

1. $\partial_\mu \varphi^a \succ \varphi^a$
2. $\partial_\mu \varphi^a \succ \partial_\nu \varphi^b \quad \Longleftrightarrow \quad \partial_\rho \partial_\mu \varphi^a \succ \partial_\rho \partial_\nu \varphi^b$

If $\boldsymbol{\alpha} \succ \boldsymbol{\beta} \Longrightarrow \partial_{\boldsymbol{\alpha}} \varphi^a \succ \partial_{\boldsymbol{\beta}} \varphi^b$ the ranking is orderly.
If $a \succ b \Longrightarrow \partial_{\boldsymbol{\alpha}} \varphi^a \succ \partial_{\boldsymbol{\beta}} \varphi^b$ the ranking is elimination.

Differential Polynomials

Let system

$$F = \{\ f_j(x_i, u^\alpha, \dots, u^\alpha_\mu) \mid 1 \le i \le n,\ 1 \le j \le k,\ 1 \le \alpha \le m\ \}$$

be a set of differential polynomials, i.e. polynomials in u^α and its derivatives, over a zero characteristic differential (coefficient) field $\mathbb{K}$, and $\succ$ be a ranking. Then every element $f \in F$ is a polynomial in its highest ranking partial derivative (leader) $\mathrm{ld}(f)$

$$f = a_0\,\mathrm{ld}(f)^d + a_1\,\mathrm{ld}(f)^{d-1} + \cdots + a_d$$

$0 \ne a_0$ is initial of f $(\mathrm{init}(f))$ and $\partial_{\mathrm{ld}(f)} f$ is separant of f $(\mathrm{sep}(f))$.

u-conditions

Definition

Given a system S of PDEs, and a ranking $\succ$, we shall say that $\mathrm{order}_{\succ}(S) = u$ if

$$u = \max_{\succ} \{ \mathrm{ld}(p) \mid p \in S \}$$

Definition

Given a system S of PDEs with $\mathrm{order}_{\succ}(S) = u$, its differential consequence p will be called *u-condition* to S if $\mathrm{ld}(p) \prec u$.

An involutive completion algorithm exploits ranking to detect all such conditions and incorporate them into system. If ranking $\succ$ is graded orderly, then u-condition is an integrability condition.

Lagrangian constraints as u-conditions

Consider again field equations for $\mathcal{L}(\varphi, \partial\varphi)$, $\partial_\mu \equiv \partial_{x_\mu}$, $(x_0 \equiv t)$

$$S := \begin{cases} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} - \frac{\partial \mathcal{L}}{\partial \varphi^a} = 0 \\ a = 1, \ldots, m \end{cases}$$

and choose the orderly ∂_t-elimination ranking $\succ$ s.t. for all a, b and nonnegative integers i_k $(k = 1, 2, 3)$

$$\partial_t \varphi^a \succ \frac{\partial^{i_1+i_2+i_3}}{\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \partial_{x_3}^{i_3}} \varphi^b$$

Proposition

Let $u := \partial_t^2 \psi$ where $\psi := \min_\succ \{\varphi^a \mid a \in \{1, \ldots, m\}\}$. Then u-condition to S is a (generalized) Lagrangian constraint.

Differential systems

Definition

Let $S^{=}$ and $S^{\neq}$ be finite sets of differential polynomials such that $S^{=} \neq \emptyset$ and contains equations

$$(\forall s \in S^{=})\ \ [\, s = 0\,]$$

whereas $S^{\neq}$ contains inequations

$$(\forall s \in S^{\neq})\ \ [\, s \neq 0\,]$$

Then the pair $(S^{=}, S^{\neq})$ of sets $S^{=}$ and $S^{\neq}$ is differential system.

Denote by $\mathfrak{Sol}(S^{=}/S^{\neq})$ the set of common solutions to $\{\, s = 0 \mid s \in S^{=}\}$ that do not annihilate $s \in S^{\neq}$.

Algebraically simple systems

Definition

A differential system $S = (S^{=}, S^{\neq})$ is said to be algebraically simple (with respect to $\succ$), if the following three conditions are satisfied, where $S_{\prec v}$ is the subsystem of S consisting of those equations and inequations whose leader is ranked lower than the variable v.

1. All $p_i \in S^{=}$ and all $q_j \in S^{\neq}$ are non-constant polynomials.
2. The leaders of all $p_i = 0$ and $q_j \neq 0$ are pairwise distinct.
3. If v is the leader of $p_i = 0$ or $q_j \neq 0$, then neither the initial nor the discriminant of that equation or inequation has a solution (over the complex numbers) in common with the subsystem $S_{\prec v}$.

Differentially simple systems

Definition

A differential system $S = \left(S^{=}, S^{\neq}\right)$ is said to be (differentially) simple (with respect to $\succ$), if the following three conditions are satisfied.

1. The system S is algebraically simple (with respect to $\succ$).
2. $S^{=}$ is involutive and minimal (as involutive basis of the ideal it generates).
3. The left hand side of every inequation $q_j \in S^{\neq}$ is reduced modulo the left hand sides of the equations in $S^{=}$, in the sense that no pseudo-division of $q_j \in S^{\neq}$ modulo any $p_i \in S^{=}$ is possible.

Decomposition into differentially simple subsystems

Theorem (Thomas' decomposition)

Any differential system $\left(S^{=}, S^{\neq}\right)$ can be decomposed into a finite set of simple subsystems $\left(S_i^{=}, S_i^{\neq}\right)$ with disjoint set of solutions

$$(S^{=}, S^{\neq}) \Longrightarrow \bigcup_i (S_i^{=}, S_i^{\neq}), \qquad \mathfrak{Sol}(S^{=}, S^{\neq}) = \biguplus_i \mathfrak{Sol}(S_i^{=}, S_i^{\neq})$$

Given such a decomposition, one can algorithmically verify if a differential equation is a consequence of the system $(S^{=}, S^{\neq})$

$$(\forall a \in \mathfrak{Sol}(S^{=}, S^{\neq}))\ [f(a) = 0] \Longleftrightarrow (\forall\, i\,)\ [\mathrm{dprem}_{\mathcal{J}}(f, S_i^{=}) = 0]$$

where $\mathrm{dprem}_{\mathcal{J}}(f, P)$ denotes differential Janet pseudo-reminder of f modulo P which is computed in the package.

Sylvester Matrix

Let differential polynomials f and g have the same leader x

$$f = \sum_{i=0}^{m} a_i x^{m-i}, \; g = \sum_{j=0}^{k} b_j x^{k-j}, \; m, k \in \mathbb{N}, \; a_0 b_0 \neq 0.$$

Then the Sylvester matrix $\mathcal{M}(f, g)$ reads

$$\mathcal{M}(f,g) = \begin{pmatrix} a_0 & a_1 & \cdots & a_m & & & & \\ & a_0 & a_1 & \cdots & a_m & & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_0 & a_1 & \cdots & a_m \\ & & & b_0 & b_1 & \cdots & b_k \\ & & b_0 & b_1 & \cdots & b_k & & \\ & \cdots & \cdots & \cdots & \cdots & & & \\ b_0 & b_1 & \cdots & b_k & & & & \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} k \;\; rows \\ \left.\vphantom{\begin{matrix} b \\ b \\ b \\ b \end{matrix}}\right\} m \;\; rows \end{array}$$

Resultants

The resultant of f and g denoted by $\mathcal{R}_0(f,g)$ is $\det[\mathcal{M}(f,g)]$. The $\rho-$th principal resultant $\mathcal{R}_\rho(f,g)$ $(\rho > 0)$ is the determinant of matrix obtained from $\mathcal{M}(f,g)$ by deleting the first and last ρ columns and the first and last ρ rows. If after the deletion the matrix becomes empty we define $\mathcal{R}_\rho(f,g) = 1$.

Theorem. (Thomas'37)

. Let R be a unique factorization domain with identity. Then

1. f and g have a common factor (greatest common divisor) $h \in R[x]$ of degree d iff $\mathcal{R}_0(f,g) = \mathcal{R}_1(f,g) = \cdots = \mathcal{R}_{d-1}(f,g) = 0$ and $\mathcal{R}_d(f,g) \neq 0$.
2. Unless $k = m = d$ there exist unique $f_1, g_1 \in R[x]$ such that $\mathcal{R}_d^2 f = f_1 h, \quad \mathcal{R}_d^2 g = g_1 h$.

   In the special case when $k = m = d$ (in this case $\mathcal{R}_d := 1$) any of polynomials f, g can be considered as their common factor h.

Splitting

To provide the first two simplicity conditions one does split as follows.

- Split by the initial of $f \in (S^{=}, S^{\neq})$:

$$f = a_0\, \mathrm{ld}(f)^d + \cdots \longrightarrow \begin{cases} a_0 = 0 \xrightarrow{\text{new system}} (S^{=} \cup \{a_0\}, S^{\neq}) \\ a_0 \neq \mathbb{K} \longrightarrow (S^{=}, S^{\neq} \cup \{a_0\}) \end{cases}$$

- Split by the ρ-th discriminant $\mathcal{D}_\rho(f) := \mathcal{R}_\rho(f, f'_{\mathrm{ld}(f)})$, $f \in (S^{=}, S^{\neq})$ if $\mathcal{D}_0(f) = \cdots = \mathcal{D}_{\rho-1}(f) = 0$ and $\mathcal{D}_\rho(f) \neq 0$:

$$\begin{cases} \mathcal{D}_\rho(f) = 0 \xrightarrow{\text{new system}} (S^{=} \cup \{\mathcal{D}_\rho(f)\}, S^{\neq}) \\ \mathcal{D}_\rho(f) \neq \mathbb{K} \longrightarrow (S^{=}, S^{\neq} \cup \{\mathcal{D}_\rho(f)\}) \mid_{f:=f_1} \end{cases}$$

Triangularization and Consistency Check

If there are two elements f, g in a system $(S^=, S^{\neq})$ with the same leader, then we compute their common factor h and co-factors f_1, g_1 given in the above Thomas theorem.

Then

1. $$f, g \in S^= \longrightarrow \begin{cases} h \in \mathbb{K} \rightarrow \text{ inconsistency} \\ h \notin \mathbb{K} \rightarrow S^= = S^= \setminus \{f, g\} \cup \{h\} \end{cases}$$

2. $f \in S^= \wedge g \in S^{\neq} \rightarrow$
$$\begin{cases} h \in \mathbb{K} \rightarrow S^{\neq} = S^{\neq} \setminus \{g\} \\ h \notin \mathbb{K} \rightarrow \begin{cases} f_1 \in \mathbb{K} \rightarrow \text{ inconsistency} \\ f_1 \notin \mathbb{K} \rightarrow S^= = S^= \setminus \{f\} \cup \{f_1\},\ S^{\neq} = S^{\neq} \setminus \{g\} \cup \{g_1\} \end{cases} \end{cases}$$

3. $f, g \in Q \rightarrow S^{\neq} = S^{\neq} \setminus \{f, g\} \cup \{f_1 g_1 h\} \leftarrow \mathrm{lcm}(f, g)$

Computation of Lagrangian constraints: algorithm

1. Input: $\begin{cases} \text{system } S \text{ of Euler-Lagrange equations} \\ \partial_t - \text{elimination ranking } \succ \\ \psi := \min_{\succ}\{\varphi^a \mid a \in \{1,\ldots,m\}\} \end{cases}$
2. Compute Thomas decomposition

$$S \Longrightarrow \bigcup_i (S_i^{=}/S_i^{\neq})$$

3. From each $S_i^{=}$ extract the set C_i of Lagrangean constraints

$$C_i := \{s \in S_i^{=} \mid \mathrm{ld}(s) \prec \partial_t^2 \psi\}$$

4. Output: $\bigcup_i (C_i/S_i^{\neq})$

Pros and cons

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  - √ The procedure is fully algorithmic.
  - √ The rank dependence of Hessian on $(\varphi, \partial\varphi)$ or $(q, \dot{q})$ is automatically taken into account.
  - √ Algebraic independence of the output constraints.
  - √ Thomas decomposition algorithm has been implemented in Maple and the code is available on the Web page http://wwwb.math.rwth-aachen.de/thomasdecomposition/index.php
  - √ Each output subsystem algorithmically admits well-posedness of Cauchy problem.
- Cons

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  - √ Each output subsystem algorithmically admits well-posedness of Cauchy problem.
- Cons
  - √ Thomas decomposition for ∂_t-elimination ranking computationally may be very costly.

(1+1)-dimensional chiral Schwinger model (Das,Ghosh'2009) I

$$\mathcal{L} = \frac{1}{2}(\partial_t A_0 - \partial_x A_1)^2 + \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_1 \phi)^2 + e\,(\partial_t \phi)\,A_0 + e\,\phi(\partial_t\,A_1) + e\,(\partial_1 \phi)\,(A_0 - A_1) + \frac{1}{2}\,a\,e^2\,(A_0^2 - A_1^2)\,.$$

Here, e, a are parameters, t, x are independent variables and $\varphi^1 = A_0$, $\varphi^2 = A_1$, $\varphi^3 = \phi$ are dependent variables. The Euler-Lagrange equations:

$$\begin{cases} \underline{\partial_t^2 A_0} - \partial_t \partial_x A_1 - e\,(\partial_t \phi + \partial_x \phi) - a\,e^2\,A_0 = 0\,, \\ \underline{\partial_t \partial_x A_0} - e\,(\partial_t \phi + \partial_x \phi) - \partial_x^2 A_1 - a\,e^2\,A_1 = 0\,, \\ \underline{\partial_t^2 \phi} + e\,(\partial_t A_0 - \partial_t A_1) - \partial_x^2 \phi + e\,(\partial_x A_0 - \partial_x A_1) = 0\,. \end{cases}$$

(1+1)-dimensional chiral Schwinger model (Das,Ghosh'2009) II

Hessian $H = \mathrm{diag}(1,0,1)$. Hence, the model is singular. We choose the following ranking:

$$w \prec v \prec u \prec w_x \prec v_x \prec u_x \prec w_{x,x} \prec \ldots \prec w_t \prec v_t \prec u_t \prec w_{t,x} \prec v_{t,x} \prec u_{t,x} \prec w_{t,x,x} \prec \ldots$$

The Euler-Lagrange equations are linear. In this case Thomas' decomposition just complete them to involution without splitting:

$$\mathbf{(1-a)\,\partial_t A_0 + (1+a)\,\partial_x A_0 - \partial_t A_1 - \partial_x A_1 = 0}\,,$$
$$(1+a)\,(\partial_t^2 A_1 - \partial_x^2 A_0) - 2e(1+a)(\partial_t\phi + \partial_x\phi) - ae^2(A_0 + A_1) - a^2e^2A_1 = 0\,,$$
$$(a+1)(\partial_t\partial_x A_1 - \partial_x^2 A_0) - e\,(\partial_t\phi + \partial_x\phi) + a\,e^2\,A_1 = 0\,,$$
$$\partial_t^2\phi - \partial_x^2\phi - e\,a\,(\partial_t A_1 - \partial_x A_0) = 0\,.$$

The first equation is a Lagrangian constraints.

Dynamical system (Deriglazov'2010, Eq.8.1) I

$$L = q_2^2\,(q_1)_t^2 + q_1^2\,(q_2)_t^2 + 2\,q_1\,q_2\,(q_1)_t\,(q_2)_t + q_1^2 + q_2^2$$

We choose the ranking $\succ$ such that

$$q_2 \prec q_1 \prec (q_2)_t \prec (q_1)_t \prec (q_2)_{t,t} \prec (q_1)_{t,t} \prec \dots$$

Euler-Lagrange equations (with underlined leaders) are

$$\begin{cases} 4\,q_2\,(q_2)_t\,(q_1)_t + 2\,q_2^2\,\underline{(q_1)_{t,t}} + 2\,q_1\,q_2\,(q_2)_{t,t} - 2\,q_1 & = & 0 \\ 4\,q_1\,(q_2)_t\,(q_1)_t + 2\,q_1^2\,(q_2)_{t,t} + 2\,q_1\,q_2\,\underline{(q_1)_{t,t}} - 2\,q_2 & = & 0 \end{cases}$$

and Hessian

$$H^{(1)} = \begin{pmatrix} 2\,q_2^2 & 2\,q_1\,q_2 \\ 2\,q_1\,q_2 & 2\,q_1^2 \end{pmatrix}.$$

Dynamical system (Deriglazov'2010, Eq.8.1) II

Thomas' decomposition produces 3 differentially simple system

$$(T_1)\left\{\begin{array}{rcl} 2\,q_2\,(q_2)_{t,t}+2\,(q_2)_t^2-1 & = & 0, \\ \boldsymbol{q_1-q_2} & \boldsymbol{=} & \boldsymbol{0}, \\ q_2 & \neq & 0 \end{array}\right.$$

$$(T_2)\left\{\begin{array}{rcl} 2\,q_2\,(q_2)_{t,t}+2\,(q_2)_t^2-1 & = & 0, \\ \boldsymbol{q_1+q_2} & \boldsymbol{=} & \boldsymbol{0}, \\ q_2 & \neq & 0 \end{array}\right.$$

$$(T_2)\left\{\begin{array}{rcl} q_1 & = & 0, \\ q_2 & = & 0. \end{array}\right.$$

The local Lagrangian constraints in the simple systems (T_1) and (T_2) can be combined in a single global constraint $\boldsymbol{q_1^2-q_2^2=0}$.

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