Local integrability and linearizability of a $(1 : -1 : -1)$ resonant quadratic system

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- 2−dimensional systems:
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  - cubic sytems: integrability and linearizability - some families of the systems studied; open problem
  - $(p:-q)$ resonant singularities - some researches

- 3−dimensional systems (integrability and linearizability not studied in such extension):
  - most studied Lotka-Volterra systems
  - Darboux integrability of three dimensional systems
  - quadratic systems in a neighborhood of a $(0:-1:1)$ resonant singular point (Z.Hu, M. Han, V.G. Romanovski)
  - recent studies:
    - W. Aziz, C.Christopher [2]: Lotka-Volterra quadratic systems with a $(1;-1;1)$,$(2;-1;1)$ and $(1;-2;1)$-resonant points
    - W. Aziz [3]: particular family of quadratic systems with a $(1:-1:1)$ resonant singularity

$$\begin{aligned} \dot{x} &= \lambda_1 x + X_1(x,y,z) = P(x,y,z) \\ \dot{y} &= \lambda_2 y + X_2(x,y,z) = Q(x,y,z) \\ \dot{z} &= \lambda_3 z + X_3(x,y,z) = R(x,y,z), \qquad \lambda_1,\lambda_2,\lambda_3 \neq 0 \end{aligned} \tag{1}$$

Formal Normal Form Theorem $\Rightarrow$ exists a change of coordinates:

$$Y = X + H(X), \tag{2}$$

where $Y = (x_1, y_1, z_1)$, $X = (x, y, z)$ and $H(X)$ is a series that does not contain linear terms, which transforms (1) to

$$\begin{aligned} \dot{x_1} &= \lambda_1 x_1 + Z_1(x_1,y_1,z_1) \\ \dot{y_1} &= \lambda_2 y_1 + Z_2(x_1,y_1,z_1) \\ \dot{z_1} &= \lambda_3 z_1 + Z_3(x_1,y_1,z_1), \end{aligned} \tag{3}$$

where $Z_i(x_1, y_1, z_1)$ has every nonresonant term equal to zero for $i = 1, 2, 3$.

Resonant terms

Monomial g_k, $k = 1, \ldots, n$, of the form $g^{(\alpha)} y^{\alpha} e_k$ with

$$< \lambda, \alpha > - \lambda_k = 0$$

System $\dot{x} = Ax + f(x)$ is *locally integrable* if it has $n-1$ functionally independent analytic first integrals in a neighbourhood of the origin.

Definition

System (1) is *locally integrable*: exists a transformation of variables (2) that transforms systems (1) into systems of the form

$$\begin{aligned} \dot{x_1} &= \lambda_1 x_1 (1 + O(x_1, y_1, z_1)) \\ \dot{y_1} &= \lambda_2 y_1 (1 + O(x_1, y_1, z_1)) \\ \dot{z_1} &= \lambda_3 z_1 (1 + O(x_1, y_1, z_1)). \end{aligned} \tag{4}$$

System (4) has analytic first integrals: by $\phi_1(x_1, y_1, z_1) = x_1^{-\lambda_2} y_1^{\lambda_1}$ and $\psi_1(x_1, y_1, z_1) = x_1^{-\lambda_3} z_1^{\lambda_1} \Rightarrow$
$\phi(x, y, z) = x^{-\lambda_2} y^{\lambda_1}(1 + O(x, y, z))$ and
$\psi(x, y, z) = x^{-\lambda_3} z^{\lambda_1}(1 + O(x, y, z))$ (two independent first integrals of systems (1))

Theorem[V.G. Romanovski, Y. Xia, X. Zhang (2014)]

$$\dot{x} = Ax + f(x) \tag{5}$$

(a) There exist series $\psi(x)$ with its resonant monomials arbitrary such that:

$$\mathcal{X}(\psi(x)) = \sum_{\alpha\in\mathfrak{R}} p_\alpha x^\alpha, \tag{6}$$

where p_α are polynomials in the coefficients of (5).

(b) If the vector field (5) has $n-1$ functionally independent analytic or formal first integrals, then for any ψ satisfying (6), we have: $p_\alpha = 0,$ for all $\alpha \in \mathfrak{R}$.

(c) Assume that the rank of $\mathfrak{R} = \{\alpha \in \mathbb{Z}_+^n | <\lambda, \alpha>= 0, |\alpha| > 0\}$ is k, i.e. $r_\lambda = k$, and there are k functionally independent $\psi^{(1)}, \ldots, \psi^{(k)}$, such that for the corresponding coefficients in (6) hold $p_\alpha^{(i)} = 0,$ for all $\alpha \in \mathfrak{R}$, $i = 1, \ldots, k$. Then the vector field $\mathcal{X}$ has exactly k functionally independent analytic or formal first integrals.

$\mathcal{B} = \langle p_\alpha^{(i)} \mid \alpha \in \mathfrak{R}, \quad i = 1, \ldots, n-1 \rangle$, where $p_\alpha^{(i)}$ are *focus quantities*

$\mathbf{V}(\mathcal{B})$... the *integrability variety* of system (5)

finite number of polynomials $p_\alpha^{(s)}$: $\sqrt{\mathcal{B}_1} \subseteq \sqrt{\mathcal{B}_2} \subseteq \sqrt{\mathcal{B}_3} \subseteq \ldots$, where $\mathcal{B}_k = \langle p_\alpha^{(1)}, \ldots, p_\alpha^{(n-1)} | \alpha \in \mathfrak{R}, |\alpha| \leq k, \ k \in \mathbb{N} \rangle$, stabilizes (find m such that $\sqrt{\mathcal{B}_m} = \sqrt{\mathcal{B}_{m+1}}$) (Hilbert's theorem)

1) irreducible decomposition of $\mathbf{V}(\mathcal{B}_m)$
("solve": $p_\alpha^{(s)} = 0$, $\alpha \in \mathfrak{R}$, $s = 1, \ldots, n-1$),
2) different methods (Darboux method) to show $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_m)$
(all systems corresponding to points from $\mathbf{V}(\mathcal{B}_m)$ have $n-1$ functionally independent analytic or formal first integrals)

Definition

A **Darboux factor** for the vector field $\mathcal{X}$ is a polynomial $f(x,y,z)$, such that

$$\mathcal{X}f = Kf, \tag{7}$$

where $K(x,y,z)$ is also a polynomial function called **cofactor** of f.
A **exponential factor** is as a function of the form

$$E(x,y,z) = e^{\frac{g(x,y,z)}{h(x,y,z)}}$$

such that g and h are coprime and $\mathcal{X}E = C_E E$ holds for some polynomial function C_E of degree at most $d-1$
$(d = \max \text{degree}(P,Q,R))$

Theorem

Suppose that $\mathbb{C}$-polynomial systems (1) *of degree d admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$ and q exponential factors $exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$. There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$, if and only if the (multivalued) function*

$$H(x, y, z) = f_1^{\lambda_1} \ldots f_p^{\lambda_p} \left(exp\left(g_1/h_1\right)\right)^{\mu_1} \ldots \left(exp\left(g_q/h_q\right)\right)^{\mu_q} \quad (8)$$

is a nontrivial first integral of systems (1) **(Darboux first integral)**.

Definition

A function M is called an inverse **Jacobi multiplier** for the vector field $\mathcal{X}$ if it satisfies the equation

$$\mathcal{X}(M) = Mdiv(\mathcal{X}) \Leftrightarrow div(\mathcal{X}/M) = 0. \quad (9)$$

A *Darboux inverse Jacobi Multiplier* D must satisfy

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = div\mathcal{X} = P_x + Q_y + R_z.$$

inverse Jacobi multiplier + one first integral $\Rightarrow$ the second first integral

Definition

System (1) is *linearizable*: exists a transformation of variables (2) which transforms systems (1) into systems

$$\begin{aligned}\dot{x}_1 &= \lambda_1 x_1\\ \dot{y}_1 &= \lambda_2 y_1\\ \dot{z}_1 &= \lambda_3 z_1.\end{aligned} \tag{10}$$

Darboux linearizability

Definition

Darboux linearization: analytic change of variables

$$x_1 = Y_1(x,y,z), \quad y_1 = Y_2(x,y,z), \quad z_1 = Y_3(x,y,z),$$

whose inverse linearizes systems (1) and such that $Y_1(x,y,z)$, $Y_2(x,y,z)$ and $Y_3(x,y,z)$ are of the form

$$Y_1(x,y,z) = \prod_{j=0}^{m} f_j^{\alpha_j}(x,y,z) = x + Y_1'(x,y,z),$$

$$Y_2(x,y,z) = \prod_{j=0}^{m} g_j^{\beta_j}(x,y,z) = y + Y_2'(x,y,z),$$

$$Y_3(x,y,z) = \prod_{j=0}^{m} h_j^{\gamma_j}(x,y,z) = z + Y_3'(x,y,z),$$

generalized Darboux linearization transformation such that $Y_1(x,y,z)$, $Y_2(x,y,z)$ and $Y_3(x,y,z)$ are Darboux functions

Theorem

The system (1) is Darboux linearizable $\iff \exists\ s+1 \geq 1$ *algebraic Darboux factors* $f_0, \ldots, f_s(K_0, \ldots, K_s)$, $t+1 \geq 1$ *algebraic Darboux factors* $g_0, \ldots, g_t(L_0, \ldots, L_t)$, *and* $u+1 \geq 1$ *algebraic Darboux factors* $h_0, \ldots, h_u(M_0, \ldots, M_u)$*:*

(1) $f_0(x,y,z) = x + \ldots$ *but* $f_j(0,0,0) = 1$ *for* $j \geq 1$;

(2) $g_0(x,y,z) = y + \ldots$ *but* $g_j(0,0,0) = 1$ *for* $j \geq 1$;

(3) $h_0(x,y,z) = z + \ldots$ *but* $h_j(0,0,0) = 1$ *for* $j \geq 1$; *and*

(4) *there are* $s+t+u$ *constants* α_i, β_j, γ_k, $i = 1, ..., s$, $j = 1, ..., t$ *and* $k = 1, ..., u$*:*

$$K_0 + \alpha_1 K_1 + \ldots + \alpha_s K_s = \lambda_1$$
$$L_0 + \beta_1 L_1 + \ldots + \beta_t L_t = \lambda_2$$
$$M_0 + \gamma_1 M_1 + \ldots + \gamma_u M_u = \lambda_3.$$

The Darboux linearization of systems (1) is given by

$$x_1 = f_0 f_1^{\alpha_1} \ldots f_s^{\alpha_s}, \quad y_1 = g_0 g_1^{\beta_1} \ldots g_t^{\beta_t} \text{ and } z_1 = h_0 h_1^{\gamma_1} \ldots h_u^{\gamma_u}.$$

INTEGRABILITY and **LINEARIZABILITY**
of the system

$$\begin{aligned} \dot{x} &= \quad x + a_{12}xy + a_{13}xz + a_{23}yz = P_1(x,y,z) \\ \dot{y} &= -y + b_{12}xy + b_{13}xz + b_{23}yz = Q_1(x,y,z) \\ \dot{z} &= -z + c_{12}xy + c_{13}xz + c_{23}yz = R_1(x,y,z). \end{aligned} \tag{11}$$

Theorem (Integrability)

The quadratic three dimensional system (11) *is locally integrable if and only if one of the following conditions is satisfied*

(1) $c_{13}=c_{12}=b_{13}=b_{12}=0;$

(2) $c_{23}=c_{12}=b_{23}=b_{13}=b_{12}+c_{13}=a_{13}=a_{12}=0;$

(3) $c_{12}=b_{13}=a_{23}=a_{12}b_{23}+a_{13}c_{23}+b_{23}c_{23}=0,$
$a_{13}b_{12}-a_{13}c_{13}-b_{23}c_{13}=a_{12}b_{12}-a_{12}c_{13}+b_{12}c_{23}=0;$

(4) $c_{23}=c_{13}=c_{12}=b_{12}=a_{13}-b_{23}=a_{12}=0;$

(5) $c_{23}=b_{23}=a_{23}=a_{13}=a_{12}=0;$

(6) $c_{13}=b_{23}=b_{13}=b_{12}=a_{13}=a_{12}-c_{23}=0.$

$\Phi(x,y,z) = xy(1+O(x,y,z))$ and $\psi(x,y,z) = xz(1+O(x,y,z))$

- computed the first 11 focus quantities of $\mathcal{X}\Phi(x,y,z)$, f_i, and $\mathcal{X}\psi(x,y,z)$, g_j:

$$
\begin{aligned}
f_1 =& (a_{13}+b_{23})c_{12};\\
f_2 =& -2a_{13}b_{12} - b_{12}b_{23} + b_{12}(a_{13}+b_{23}) + (a_{13}+b_{23})c_{13}+\\
& -b_{13}c_{23};\\
f_3 =& -2a_{13}b_{13} + b_{13}(a_{13}+b_{23}); \ldots\\
g_1 =& -2a_{12}c_{12} + c_{12}(a_{12}+c_{23});\\
g_2 =& -(b_{23}c_{12}) - 2a_{12}c_{13} - c_{13}c_{23} + b_{12}(a_{12}+c_{23})+\\
& +c_{13}(a_{12}+c_{23});\\
g_3 =& b_{13}(a_{12}+c_{23}); \ldots
\end{aligned}
$$

- not unique focus quantities (free coefficients equal to zero)
- irreducible decomposition (Singular routine `minAssGTZ`); $\mathcal{B}_{11} = \langle f_1, f_2, f_3, \ldots, f_{11}, g_1, g_2, g_3, \ldots, g_{11} \rangle$

Darboux integrability

Case 2

$$\begin{aligned}\dot{x} &= x + a_{23}yz\\ \dot{y} &= -y(1-b_{12}x)\\ \dot{z} &= -z(1+b_{12}x)\end{aligned}$$

Darboux factors: $l_1 = y$, $l_2 = z$, $l_3 = x + \frac{a_{23}}{3}yz$
Exponential factor: $l_4 = e^{2x+a_{23}yz}$
First integrals:

$$\begin{aligned}\phi(x,y,z) &= l_2 l_3 l_4^{\frac{b_{12}}{2}} = xz(1+O(x,y,z))\\ \psi(x,y,z) &= l_1 l_2 l_3^2\end{aligned}$$

ψ (not in the wanted form) $\to (\psi/\phi)(x,y,z) = xy(1+O(x,y,z))$

Case 3

$$\begin{aligned}\dot{x} &= x(1 + a_{12}y + a_{13}z)\\ \dot{y} &= y(-1 + b_{12}x + b_{23}z)\\ \dot{z} &= z\left(-1 + \frac{a_{13}b_{12}}{a_{13}+b_{23}}x - \frac{a_{12}b_{23}}{a_{13}+b_{23}}y\right)\end{aligned}$$

change of variables $x \mapsto y, y \mapsto x, z \mapsto z$ and time rescaling $\Rightarrow$ systems studied by W. Aziz, C. Christopher [2] [Theorem 4]

Prove of sufficiency of conditions:

Linearizability $\Longrightarrow$ Integrability

Theorem (Linearizability)

The system (11) *is linearizable if and only if either one of the conditions* (1),(2),(4),(5),(6) *from Theorem on Integrability is satisfied or one of two following conditions holds*

(7) $c_{12} = b_{13} = b_{12} = b_{23} = a_{23} = a_{13} = a_{12} = 0$

(8) $b_{13} = c_{13} = c_{12} = c_{23} = a_{23} = a_{13} = a_{12} = 0$

Necessary conditions

computing linearizability quantities;

computing the irreducible decomposition (Singular routine `minAssGTZ`)

Case 1: $c_{13} = c_{12} = b_{13} = b_{12} = 0$

$$\begin{aligned} \dot{x} &= x + a_{12}xy + a_{13}xz + a_{23}yz \\ \dot{y} &= y(-1 + b_{23}z) \\ \dot{z} &= z(-1 + c_{23}y) \end{aligned} \tag{12}$$

$\dot{y}, \dot{z} \ldots$ linearizable node $\Rightarrow$ transformation $\Rightarrow \dot{Y} = -Y, \dot{Z} = -Z$
looking for $X = \alpha(Y, Z) + \beta(Y, Z)x$ such that $\dot{X} = X \Rightarrow$

$$\dot{\alpha} + \beta a_{23}yz = \alpha, \quad \dot{\beta} + \beta(a_{12}y + a_{13}z) = 0 \tag{13}$$

Case 2: $c_{23} = c_{12} = b_{23} = b_{13} = b_{12} + c_{13} = a_{13} = a_{12} = 0$

$$X = x + \frac{a_{23}}{3} yz$$

$$Y = ye^{-b_{12}x - \frac{a_{23}b_{12}}{2} yz}$$

$$Z = ze^{b_{12}x + \frac{a_{23}b_{12}}{2} yz}$$

Transformation of cases

Case 4 $\Leftrightarrow$ Case 6
Case 7 $\Leftrightarrow$ Case 8

Remark

some invariant curves obtained from first integrals (proof of integrability)

Theorem (First integral $xy(1+O(x,y,z))$)

Necessary conditions for three dimensional system (11) with $a_{23}=0$ to have a first integral in the form $xy(1+O(x,y,z))$ are

(1) $c_{12}=b_{13}=a_{13}b_{12}-a_{13}c_{13}-b_{23}c_{13}=0;$

(2) $c_{13}=c_{12}=a_{13}-b_{23}=b_{12}b_{23}+b_{13}c_{23}=0;$

$\vdots$

(7) $c_{23}=b_{23}=a_{13}=0;$

(8) $b_{23}=b_{13}=a_{13}=0;$

(9) $c_{13}=b_{13}=b_{12}=a_{13}+b_{23}=0;$

(10) $b_{13}=b_{12}=a_{13}+b_{23}=a_{12}=0;$

(11) $b_{13}=b_{12}=a_{13}+b_{23}=b_{23}c_{12}+a_{12}c_{13}=0;$

(12) $b_{13}=b_{12}=a_{13}+b_{23}=b_{23}c_{12}+c_{13}c_{23}=0;$

Computation of the necessary conditions for existence one first integral in the form $xy(1+O(x,y,z))$
similar way as for Theorem (8)
computed first few focus quantities, appearing free coefficients-resonant coefficients (denoted as $p1, p2, p3, \ldots$)
first three focus quantities (f_1, f_2, f_3) were presented before and in f_4 appear free conditions

$$\begin{aligned} f_4 &= (b_{12}(9a_{12}^2 b_{12} + 6a_{12}a_{13}c_{12} - a_{23}b_{12}c_{12} + 9a_{12}b_{23}c_{12} + \\ &+ a_{23}c_{12}c_{13} + 3a_{13}c_{12}c_{23} + 3b_{23}c_{12}c_{23} + 12a_{12}p1))/2 + \ldots \end{aligned}$$

eliminating free coefficients from first thirteen focus quantities
(Singular routine `Eliminate`- laborious routine)
Elimination Theorem
Eliminate: geometrically: projection P of the variety of ideal $F = \langle f_1, \ldots, f_{13} \rangle$ on the space of parameters
computing irreducible decomposition (`minAssGTZ`)
$\Rightarrow$ twelve cases
Computational problems: difficult to compute over the field of rational numbers $\Rightarrow a_{23} = 0$;
modular arithmetics [4]$\Rightarrow$ one condition incorrect
$(5') \quad c_{12} = 152b_{12} + 13c_{13} = a_{13} - b_{23} = 39a_{12} + 7c_{23} = 71b_{23}c_{13} - 135b_{13}c_{23} = 0$ (corrected: adding $c_{13}b_{23} = 0$)

Remark 1

- used modular arithmetics $\Rightarrow$ can not guarantee that the list of conditions is complete

Remark 2

- proof of sufficiency done only for three cases: Darboux integrability

Remark 3

- complete integrability: resonant coefficients all zero
- one first integral: resonant coefficient can not be equal to zero
  Explanation:
  - resonant coefficients equal to zero; minimal decomposition: 7 conditions $(J_1,\ldots,J_7)$
  - resonant coefficient free; first elimination of res. coef., then minimal decomposition: 12 conditions $(I_1,\ldots,I_{12})$
  - $M_1=\cap_{i=1}^{7}J_i$, $M_2=\cap_{i=1}^{12}I_i$;
    comparison: $\texttt{Reduce}(M_1,M_2)=0 \Longleftrightarrow M_1\subset M_2$

Conclusion

$$\begin{aligned}\dot{x} &= \quad x + a_{12}xy + a_{13}xz + a_{23}yz \\ \dot{y} &= -y + b_{12}xy + b_{13}xz + b_{23}yz \\ \dot{z} &= -z + c_{12}xy + c_{13}xz + c_{23}yz\end{aligned}$$

- Two independent first integrals of (11): 6 cases
- Linearizable systems (11): 7 cases
- One first integral of (11): 12 cases ⇒ interesting observation

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Thank you for your attention!