Integrability of Polynomial Systems of Ordinary Differential Equations

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Based on the works:


Consider the system
\[ \begin{align*}
\dot{u} &= -v + p(u, v), \\
\dot{v} &= u + q(u, v),
\end{align*} \] (1)

where \( p \) and \( q \) are convergent series without free and linear terms. It has a center at the origin (all trajectories are ovals) iff it is locally analytically equivalent to a system of the form
\[ \begin{align*}
\dot{x} &= ix(1 + g(xy)), \\
\dot{y} &= -iy(1 + g(xy)),
\end{align*} \] (2)

where, \( i = \sqrt{-1} \), \( x = u + iv \) and \( y = \bar{x} \).

\[ \rightarrow \] \( xy \) is a first integral of (2)
\[ \rightarrow \] \( u^2 + v^2 + h.o.t. \) is a first integral of (1)

**Theorem (Poincaré-Lyapunov)**

*System (1) has a center at the origin iff it admits a first integral of the form \( u^2 + v^2 + h.o.t. \).*

We discuss a generalization of the center problem (the Poincaré integrability problem) to \( n \)-dim systems.

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Integrability of Polynomial Systems of ODEs
\[ \dot{x} = Ax + f(x), \]  
\[ (3) \]

A is \( n \times n \) matrix, \( x = (x_1, \ldots, x_n)^\tau \), \( f(x) = (f_1(x), \ldots, f_n(x))^\tau \), and \( f_i \) are series starting with at least quadratic terms.

Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be the \( n \)-tuple of eigenvalues of \( A \). Set \( \mathbb{Z}_+ = \mathbb{N} \cup 0 \). For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \) denote

\[ \langle \lambda, \alpha \rangle = \sum_{i=1}^{n} \alpha_i \lambda_i \]

and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). Let

\[ \mathcal{R} = \{ \alpha \in \mathbb{Z}_+^n \mid \langle \lambda, \alpha \rangle = 0, \ |\alpha| > 0 \}, \]

and denote by \( r_\lambda \) the rank of vectors in the set \( \mathcal{R} \).
A substitution

\[ x = \Phi(y) := y + \varphi(y), \]  
transforms (3) to its Poincaré–Dulac normal form, i.e. a system of the form

\[ \dot{y} = Ay + g(y), \]  
where \( g(y) = (g_1(y), \ldots, g_n(y))^\tau \) contains only resonant terms, that is, each monomial in \( g_k, k = 1, \ldots, n \), is of the form \( g^{(\alpha)}y^\alpha e_k \) with

\[ \langle \lambda, \alpha \rangle - \lambda_k = 0, \]

where \( e_k \) is the \( n \)–dimensional unit vector with its \( n \)th component equal to 1 and the others all equal to zero. We call the transformation (4) a normalization.

The normalization containing only nonresonant terms is unique. We call this normalization a distinguished normalization and term the corresponding Poincaré–Dulac normal form a distinguished normal form.
Normalization (4) does not necessarily converge, so generally speaking $\varphi$ and $g$ are formal power series.

**Poincaré domain** in $\mathbb{C}^n$ is the set of all points $(z_1, \ldots, z_n)$ such that the convex hull of the set $\{z_1, \ldots, z_n\} \subset \mathbb{C}$ does not contain the origin. Then if the vector $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues of $A$ in (3) lies in the Poincaré domain then there exists a convergent normalizing transformation.

**Theorem (C. L. Siegel)**

Suppose there exist positive constants $C > 0$ and $\nu > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| > 1$ and for all $k \in \{1, \ldots, n\}$ the inequality

$$\left| \sum_{i=1}^{n} \alpha_i \lambda_i - \lambda_k \right| \geq C|\alpha|^{-\nu}$$

(6)

holds. Then there exists a convergent transformation of (3) to normal form.
Theorem (V. A. Pliss)

Suppose that for system (3)
(i) the nonzero elements among the $\sum_{j=1}^{n} \alpha_j \lambda_j - \lambda_k$ satisfy condition (6)
(ii) some formal normal form of (3) is linear.
Then there exists a convergent transformation to normal form.

Bryuno conditions that together are sufficient for existence of a convergent normalizing transformation:

1) Condition $\omega$: for $w_\ell = \min(\alpha, \lambda)$ over all $\alpha \in \mathbb{N}_0^n$ for which $(\alpha, \lambda) \neq 0$ and $|\alpha| \leq 2^\ell$, $\sum 2^{-\ell} \ln w_\ell < \infty$;
2) Condition A (simplified version): some normal form has the form

$$\dot{y} = (1 + g(y))Ay,$$

that is, $\dot{y}_j = \lambda_j y_j (1 + g(y))$ for some scalar function $g(y)$.

Following to S. Walcher we say that (3) satisfies the Pliss-Bryuno condition if it can be transformed to (7) by a normalizing transformation.
For simplicity we assume that $A$ is in Jordan normal form and lower triangular.

**Definition**

System (3) is *(locally) analytically* (or *formally*) *integrable* if it has $n - 1$ functionally independent analytic (or formal) first integrals in a neighborhood of the origin.

**Theorem (X. Zhang, Llibre-Pantazi-Walcher)**

System (3) has $n - 1$ functionally independent analytic first integrals in a neighborhood of the origin if and only if $r_{\lambda} = n - 1$ and the distinguished normal form of (3) satisfies the Pliss-Bruno condition.
\[ \dot{u} = -v + P(u, v, w) = \tilde{P}(u, v, w) \]
\[ \lambda \in \mathbb{R} \setminus \{0\} \quad (8) \]
\[ \dot{v} = u + Q(u, v, w) = \tilde{Q}(u, v, w) \]
\[ \dot{w} = -\lambda w + R(u, v, w) = \tilde{R}(u, v, w) \]

\( P, Q, \) and \( R \) are real analytic in a neighborhood of the origin.

We look for a function \( \Phi(u, v, w) \) with undetermined coefficients \( \phi_{jkl} \),

\[ \Phi(u, v, w) = u^2 + v^2 + \sum_{j+k+\ell=3} \phi_{jkl} u^j v^k w^\ell, \quad (9) \]

such that

\[ \frac{\partial \Phi}{\partial u} \tilde{P} + \frac{\partial \Phi}{\partial v} \tilde{Q} + \frac{\partial \Phi}{\partial w} \tilde{R} \equiv 0. \quad (10) \]

Obstacles for the fulfillment of (10) will give us the necessary conditions for the existence of a first integral of the form

\[ \Phi(u, v, w) = u^2 + v^2 + \ldots. \quad (11) \]

A computational procedure to find the first \( m - 1 \) conditions for integrability is as follows.
• Write down the initial string of (9) up to order $2m$,
\[ \Phi_{2m}(u, v, w) = u^2 + v^2 + \sum_{j+k+\ell=3}^{2m} \phi_{jk\ell} u^j v^k w^\ell. \]
• For each $i = 3, \ldots, 2m + 1$ equate coefficients of terms of order $i$ in the expression
\[ \frac{\partial \Phi_{2m}}{\partial u} \tilde{P} + \frac{\partial \Phi_{2m}}{\partial v} \tilde{Q} + \frac{\partial \Phi_{2m}}{\partial w} \tilde{R} - g_1(u^2 + v^2)^2 - \cdots - g_{m-1}(u^2 + v^2)^m \] (12)
to zero obtaining $2m - 2$ systems of linear variables in unknown variables $\phi_{jk\ell}$.
Computing in this way one obtains a list of polynomials, \( g_1, g_2, g_3, \ldots \) in parameters of system (8). We call the polynomial \( g_i \) the \emph{i-th focus quantity (Lyapunov number)}. Each polynomial \( g_i \) represents an obstacle for existing of integral (9), that is, system (8) admits an integral (11) iff

\[
g_1 = g_2 = g_3 = \cdots = 0.
\]

The set of systems with a first integral of the form (11) is the set of common zeros of an infinite system of polynomials

\[
g_1 = g_2 = g_3 = \cdots = 0. \tag{13}
\]

Conditions (13) are \emph{the necessary conditions} for existence of first integral \( \Phi(u, v, w) = u^2 + v^2 + \ldots \) in system (8).
Two difficulties in computing the necessary conditions for integrability:

1) Polynomials $g_1, g_2, g_3, \ldots$ are not uniquely defined (depend on the choice of resonant terms).
Let $\mathcal{X}$ be the vector field associated to system (3).
Let $\psi(x)$ be a series. We call the term $\psi^{(\alpha)} x^\alpha$ a resonant term if $\alpha \in \mathbb{R}$ ($\langle \alpha, \lambda \rangle = 0$).

2) Solving even a finite system of polynomials

$$g_1 = g_2 = g_3 = \cdots = g_k = 0$$

can be an extremely laborious problem.
Theorem (VR, Y. Xia, X. Zhang, J. Differential Equations, 2014)

For system (3) the following statements hold.

(a) There exist series $\psi(x)$ with its resonant monomials arbitrary such that

$$\mathcal{X}(\psi(x)) = \sum_{\alpha \in \mathcal{R}} p_\alpha x^\alpha,$$

where $p_\alpha$ are functions of the coefficients of (3).

(b) If the vector field (3) has $n - 1$ functionally independent analytic or formal first integrals, then for any $\psi$ satisfying (14), we have

$$p_\alpha = 0, \quad \text{for all} \quad \alpha \in \mathcal{R}.$$

(c) Assume that the rank of $\mathcal{R}$ is $k$, i.e. $r_\lambda = k$, and there are $k$ functionally independent $\psi^{(1)}, \ldots, \psi^{(k)}$, such that for the corresponding coefficients in (14) hold

$$p^{(i)}_\alpha = 0, \quad \text{for all} \quad \alpha \in \mathcal{R}, \quad i = 1, \ldots, k.$$  

Then the vector field $\mathcal{X}$ has exactly $k$ functionally independent analytic or formal first integrals.
Definition

The variety of an ideal $I$ generated by $f_1(x_1, \ldots, x_n), \ldots, f_1(x_1, \ldots, x_n)$ of the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ is the set of all points in $\mathbb{F}^n$ where all polynomials of $I$ vanish. (The variety of $I$ is denoted by $\mathbf{V}(I)$).

W.l.o.g we can take $\psi^{(i)}_\alpha = 0$ for resonant $\alpha$. Then $p_\alpha$ are polynomials. Denote by $\mathcal{B}$ the ideal generated by the polynomials $p_\alpha$, for some choice of $n-1$ functionally independent functions $\psi^{(1)}, \ldots, \psi^{(n-1)}$ satisfying (14), i.e.

$$\mathcal{B} = \langle p^{(i)}_\alpha \mid \alpha \in \mathcal{R}, \quad i = 1, \ldots, n-1 \rangle.$$  

(16)

By the equivalence of (b) and (c) with $k = n-1$ the variety of $\mathcal{B}$, $\mathbf{V}(\mathcal{B})$, is the set of all points in the space of parameters of system (3), such that the corresponding systems have $n-1$ functionally independent integrals. We call $\mathbf{V}(\mathcal{B})$ the integrability variety of system (3).
To find the variety of $\mathcal{B}$ we can choose $n - 1$ linearly independent vectors from $\mathcal{R}$, let say $\alpha_1, \ldots, \alpha_{n-1} \in \mathcal{R}$. Then $x^{\alpha_1}, \ldots, x^{\alpha_k}$ are functionally independent (integrals of the system of the linear approximation) and we look for $n - 1$ functions $\psi_s(x) = x^{\alpha_s} + \text{higher order terms}$ satisfying

$$\mathcal{X}(\psi^{(s)}(x)) = \sum_{\alpha \in \mathcal{R}} p^{(s)}_{\alpha} x^{\alpha}.$$ 

In actual calculations we can find only a finite number of polynomials $p^{(s)}_{\alpha}$, so we compute few first polynomials $p^{(s)}_{\alpha}$ which generate some ideal $\mathcal{B}_m$. Then,

a) we find the irreducible decomposition of $\mathcal{V}(\mathcal{B}_m)$ (solve the polynomial system $p^{(s)}_{\alpha} = 0$),

b) using different methods we try to show that $\mathcal{V}(\mathcal{B}) = \mathcal{V}(\mathcal{B}_m)$, that is, all systems corresponding to points from $\mathcal{V}(\mathcal{B}_m)$ have $n - 1$ functionally independent analytic or formal first integrals.
To make a progress it is crucial to have an efficient approach for solving systems of polynomials of many variables:

\[ f_1(x_1, \ldots, x_n) = 0, \]
\[ \ldots \]
\[ f_m(x_1, \ldots, x_n) = 0. \]  

Let us find the variety in \( \mathbb{C}^3 \) of the ideal \( I = \langle f_1, f_2, f_3, f_4 \rangle \), where

\[ f_1 = 8x^2y^2 + 5xy^3 + 3x^3z + x^2yz, \]
\[ f_2 = x^5 + 2y^3z^2 + 13y^2z^3 + 5yz^4, \]
\[ f_3 = 8x^3 + 12y^3 + xz^2 + 3, \]
\[ f_4 = 7x^2y^4 + 18xy^3z^2 + y^3z^3. \]  

that is, the solution set of the system

\[ f_1 = 0, \quad f_2 = 0, \quad f_3 = 0, \quad f_4 = 0. \]

Under the lexicographic ordering with \( x > y > z \) a Gröbner basis for \( I \) is \( G = \{g_1, g_2, g_3\} \), where \( g_1 = x, \)
\[ g_2 = y^3 + \frac{1}{4}, \quad g_3 = z^2. \]  

\[ f_1 = f_2 = f_3 = f_4 = 0 \iff g_1 = g_2 = g_3 = 0 \]
This method ALWAYS works when the set of solution is finite: compute a Gröbner basis with respect to a lexicographic order, the basis MUST be triangular (like in Gauss row-echelon form, but with non-linear equations). We have the following computational obstacle: in the example below the following polynomial appears in the intermediate computations of the Gröbner basis:

\[ y^3 - 1735906504290451290764747182 \ldots \]  

The integer in the second term of the above polynomial contains roughly 80,000 digits.

- At least theoretically the Groebner basis theory allows to solve polynomial systems with a finite number of solutions.
In generic case the variety consists of infinitely many points.

"To solve" a polynomial system means to find a decomposition of the variety of the ideal (the zero set) into irreducible components, that is, to find a representation $V = V_1 \cup \cdots \cup V_m$, where each $V_i$ is irreducible.

Example. For $J = \langle xy, xz \rangle$, the variety of $J$ ($xy = zx = 0$) is the union of the plane $x = 0$ and the line $y = z = 0$. 
There are 3 algorithms for irreducible decompositions, all implemented in Singular:
http://www.singular.uni-kl.de.
- Gianni–Trager–Zacharias (1988) (minAssGTZ)
- Shimoyama–Yokoyama (1996) (primdecSY)
- Characteristic sets method (Wang, 1992) (minAssChar)
(the first one is implemented also in Maple)

>LIB "primdec.lib";
>ring r=0,(a20,a11,a02,a13,b31,b20,b11,b02),dp;
>poly g11=a11-b11;
>poly g22=a20*a02-b02*b20;
>poly g33=(3*a20^2*a13+8*a20*a13*b20+3*a02^2*b31
          -8*a02*b02*b31-3*a13*b20^2-3*b02^2*b31)/8;
>poly g44=(-9*a20^2*a13*b11+a11*a13*b20^2
          +9*a11*b02^2*b31-a02^2*b11*b31)/16;
>poly g55=(-9*a20^2*a13*b02*b20+a20*a02*a13*b20^2
          +9*a20*a02*b02^2*b31+18*a20*a13^2*b20*b31
          +6*a02^2*a13*b31^2-a02^2*b02*b20*b31
\[
\begin{align*}
\text{ideal} & \quad i = g_{11}, g_{22}, g_{33}, g_{44}, g_{55}; \\
& \quad \text{minAssGTZ}(i); \\
& \quad [1]: \\
& \quad \quad _[1] = a_{02} - 3b_{02} \\
& \quad \quad _[2] = a_{11} - b_{11} \\
& \quad \quad _[3] = 3a_{20} - b_{20} \\
& \quad [2]: \\
& \quad \quad _[1] = b_{11} \\
& \quad \quad _[2] = 3a_{02} + b_{02} \\
& \quad \quad _[3] = a_{11} \\
& \quad \quad _[4] = a_{20} + 3b_{20} \\
& \quad \quad _[5] = 3a_{13}b_{31} + 4b_{20}b_{02} \\
& \quad [3]: \\
& \quad \quad _[1] = a_{11} - b_{11} \\
& \quad \quad _[2] = a_{20}a_{02} - b_{20}b_{02} \\
& \quad \quad _[3] = a_{20}a_{13}b_{20} - a_{02}b_{31}b_{02} \\
& \quad \quad _[4] = a_{02}^2b_{31} - a_{13}b_{20}^2 \\
& \quad \quad _[5] = a_{20}^2a_{13} - b_{31}b_{02}^2
\end{align*}
\]
The notorious computational difficulty of the Gröbner basis calculations over the field of rational numbers is an essential obstacle for using the Gröbner basis theory for the real world applications.

**Modular calculations**: choose a prime number \( p \) and do all calculations modulo \( p \), that is, in the finite field of the characteristic \( p \) (the field \( \mathbb{Z}_p = \mathbb{Z}/p \)). The modular calculations still keep essential information on our original system and it is often possible to extract this information from the result of calculations in \( \mathbb{Z}_p \) and to obtain the exact solution of polynomial system over the field of rational numbers.
P. Wang’s algorithm for the rational reconstruction

Step 1. \( u = (u_1, u_2, u_3) := (1, 0, m), \ v = (v_1, v_2, v_3) := (1, 0, c) \)

Step 2. While \( \sqrt{m/2} \leq v_3 \)

\[ \{q := \lfloor u_3/v_3 \rfloor, \ r := u - qv, \ u := v, \ v := r\} \]

Step 3. If \( |v_2| \geq \sqrt{m/2} \) then error()

Step 4. Return \( v_3, v_2 \)

\( \lfloor \cdot \rfloor \) stands for the floor function.

Given an integer \( c \) and a prime number \( p \) the algorithm produces integers \( v_3 \) and \( v_2 \) such that \( v_3/v_2 \equiv c \pmod{p} \), that is, \( v_3 = v_2c + pt \) with some \( t \). If such a number \( v_3/v_2 \) does need not exist the algorithm returns "error()".
P. Wang's algorithm for the rational reconstruction

Step 1. $u = (u_1, u_2, u_3) := (1, 0, m), \ v = (v_1, v_2, v_3) := (1, 0, c)$

Step 2. While $\sqrt{m/2} \leq v_3$ do

\[
\{ q := \lfloor u_3/v_3 \rfloor, \ r := u - qv, \ u := v, \ v := r \}\]

Step 3. If $|v_2| \geq \sqrt{m/2}$ then error()

Step 4. Return $v_3, v_2$

$\lfloor \cdot \rfloor$ stands for the floor function.

Given an integer $c$ and a prime number $p$ the algorithm produces integers $v_3$ and $v_2$ such that $v_3/v_2 \equiv c \pmod{p}$, that is, $v_3 = v_2c + pt$ with some $t$. If such a number $v_3/v_2$ does need not exist the algorithm returns "error()".

For the discussed example computing the Gröbner basis of (18) over the field of characteristic 32003 we find $G = \{x, y^3 + 8001, z^2\}$.

Rational reconstruction yields $8001 \equiv 1/4 \pmod{32003}$. Therefore the reconstructed (lifted) Gröbner basis is $G = \{x, y^3 + 1/4, z^2\}$. 
Radical Membership Test

For a polynomial $f$ and an ideal $I = \langle f_1, \ldots, f_m \rangle$ in $k[x_1, \ldots, x_n]$, $k = \mathbb{C}$, $f$ is equal to zero on $V(I)$ if and only if the reduced Gröbner basis of the ideal $\langle 1 - wf, f_1, \ldots, f_m \rangle$ (here $w$ is a new variable) is equal to $\{1\}$.

Allows to check if zero sets of $I = \langle f_1, \ldots, f_m \rangle$ and $J = \langle h_1, \ldots, h_s \rangle$ are the same in $\mathbb{C}^n$. 
Decomposition Algorithm with Modular Arithmetics


- Choose a prime number $p$ and compute the minimal associated primes $\tilde{Q}_1, \ldots, \tilde{Q}_s$ of $I = \langle f_1, \ldots, f_s \rangle$ in $\mathbb{Z}_p[x_1, \ldots, x_n]$.

- Using the rational reconstruction algorithm lift the ideals $\tilde{Q}_i$ ($i = 1, \ldots, s$) to the ideals $Q_i$ in $\mathbb{Q}[x_1, \ldots, x_n]$.

- For each $i$ using the radical membership test check whether the original polynomials $f_1, \ldots, f_s$ vanish on the components $Q_i$ of the decomposition (on $V(Q_i)$), i.e. whether the reduced Gröbner basis of the ideal $\langle 1 - wf, Q_i \rangle$ is equal to $\{1\}$. If "yes", then go to the step 4, otherwise take another prime $p$ and go to step 1.

- Compute $Q = \cap_{i=1}^s Q_i \subset \mathbb{Q}[x_1, \ldots, x_n]$.

- Check that $\sqrt{Q} = \sqrt{I}$, i.e. $\forall g \in Q$ the reduced GB of the ideal $\langle 1 - wg, I \rangle$ is $\{1\}$ and $\forall f \in I$ the reduced GB of $\langle 1 - wf, Q \rangle$ is equal to $\{1\}$. If it is the case then $V(I) = \bigcup_{i=1}^s V(Q_i)$. If not, then choose another prime $p$ and go to Step 1.
Example: a 3-dim system

System with \((0 : -1 : 1)\) resonant point at the origin:

\[
\begin{align*}
\dot{x}_1 &= \sum_{i+j+k=2}^{m} p_{ijk} x_1^i x_2^j x_3^k = P(x), \\
\dot{x}_2 &= -x_2 + \sum_{i+j+k=2}^{m} q_{ijk} x_1^i x_2^j x_3^k = Q(x), \\
\dot{x}_3 &= x_3 + \sum_{i+j+k=2}^{m} r_{ijk} x_1^i x_2^j x_3^k = R(x),
\end{align*}
\]

(20)

\(P, Q, R\) are polynomials and \(x = (x_1, x_2, x_3) \in \mathbb{C}^3\). For system (20) \(\lambda = (0, -1, 1)\), thus, the set \(\mathcal{R}_\lambda\) is

\[
\mathcal{R} = \{\alpha \in \mathbb{N}^3_+ \mid \alpha_2 = \alpha_3\}.
\]

(21)

\[
\begin{align*}
\psi_1 &= x_1 + \sum_{|\alpha| > 1} \phi(\alpha) x^\alpha, \\
\psi_2 &= x_2 x_3 + \sum_{|\alpha| > 2} \psi(\alpha) x^\alpha.
\end{align*}
\]

(22) (23)
There are series $\psi_1(x)$ and $\psi_2(x)$,

$$\psi_1 = x_1 + \sum_{|\alpha|>1} \psi_1^{(\alpha)} x^\alpha$$

(24)

$$\psi_2 = x_2 x_3 + \sum_{|\alpha|>2} \psi_2^{(\alpha)} x^\alpha,$$

(25)

such that

$$\frac{\partial \psi_1}{\partial x_1} P + \frac{\partial \psi_1}{\partial x_2} Q + \frac{\partial \psi_1}{\partial x_3} R = \sum_{\alpha \in \mathbb{R}} g_\alpha(a, b, c) x^\alpha$$

(26)

and

$$\frac{\partial \psi_2}{\partial x_1} P + \frac{\partial \psi_2}{\partial x_2} Q + \frac{\partial \psi_2}{\partial x_3} R = \sum_{\alpha \in \mathbb{R}} h_\alpha(a, b, c) x^\alpha,$$

(27)

where $P$, $Q$, $R$ are the right hand sides of (20) and $g_\alpha$, $h_\alpha$ ($\alpha \in \mathbb{R}$) are polynomials in $(a, b, c)$. 
Denote by $\mathcal{B}$ the ideal generated by the polynomials $g_\alpha$ and $h_\alpha$, 
$\mathcal{B} = \langle g_\alpha, h_\alpha \mid \alpha \in \mathcal{R} \rangle$, and by $\mathbf{V}(\mathcal{B})$ its variety – $\mathbf{V}(\mathcal{B})$ is the integrability variety of (20).

The set of all integrable systems (20) in the space of parameters of the system is the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B}$ and it is the same for any choice of series (24) and polynomials $g_\alpha, h_\alpha (\alpha \in \mathcal{R})$ satisfying (26) and (27).

After decomposition of $\mathbf{V}(\mathcal{B})$ using the decomposition algorithm with modular arithmetic we obtain the necessary condition of integrability. The next step: prove their sufficiency.
Two main mechanisms for integrability:

- Darboux integrability
- Time-reversibility

\[ \frac{dz}{dt} = F(z) \quad (z \in \Omega), \]  

\( F : \Omega \mapsto T\Omega \) is a vector field and \( \Omega \) is a manifold.

**Definition**

A time-reversible symmetry of (28) is an invertible map \( T : \Omega \mapsto \Omega \), such that

\[ \frac{d(Tz)}{dt} = -F(Tz). \]
By Llibre, Pantazi and Walcher (2012) if a system (30) is time-reversible with respect to a linear invertible transformation which permutes $x_2$ and $x_3$ then it is integrable.

\[
\begin{align*}
\dot{x}_1 &= \sum a_{jkl} x_1^j x_2^k x_3^l, \\
\dot{x}_2 &= x_2 \sum b_{mnp} x_1^m x_2^n x_3^p, \\
\dot{x}_3 &= x_3 \sum c_{qrs} x_1^q x_2^r x_3^s.
\end{align*}
\] (30)

Let $u$, $v$, $w$ be the number of parameters of the first, the second and the third equation, respectively. By $(a, b, c)$ we denote the $(u + v + w)$-tuple of parameters of system (30).

System (30) is time-reversible if there exists an invertible matrix $T$ such that

\[
T^{-1} \circ f \circ T = -f.
\] (31)
We look for a transformation $T$ in the form

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & 1/\gamma & 0 \end{pmatrix}.$$  

(32)

(31) is satisfied for $T$ defined by (32) if and only if

$$a_{jkl} = -\gamma^{l-k}a_{jlk}, \quad b_{mnp} = -\gamma^{p-n}c_{mpn}.$$  

(33)

Denote by $k[a, b, c]$ the ring of polynomials in parameters of system (30) with the coefficients in a field $k$ and

$$H = \langle 1 - y\gamma, \ a_{jkl} + \gamma^{l-k}a_{jlk}, \ b_{mnp} + \gamma^{p-n}c_{mpn} \rangle,$$  

(34)

where $y$ is a new variable.
Suppose we are given the system of equations

\[ x_1 = \frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)}, \ldots, x_n = \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)}, \]  

(35)

where \( f_j, g_j \in k[t_1, \ldots, t_m] \) for \( j = 1, \ldots, n \). Let \( W = \mathbf{V}(g_1 \cdots g_n) \).

Equations (35) define

\[ F : k^m \setminus W \rightarrow k^n \]

by

\[ F(t_1, \ldots, t_m) = \left( \frac{f_1(t_1, \ldots, t_m)}{g_1(t_1, \ldots, t_m)}, \ldots, \frac{f_n(t_1, \ldots, t_m)}{g_n(t_1, \ldots, t_m)} \right). \]  

(36)

The image of \( k^m \setminus W \) under \( F \) denote by \( F(k^m \setminus W) \) is not necessarily an affine variety.
Consequently we look for the smallest affine variety that contains $F(k^m \setminus W)$, i.e., its Zariski closure $\overline{F(k^m \setminus W)}$. The problem of finding $\overline{F(k^m \setminus W)}$ is known as the problem of \textit{rational implicitization} (e.g. Cox et al, 2003).

**Rational implicitization theorem**

Let $k$ be an infinite field, let $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ be elements of $k[t_1, \ldots, t_m]$, let $W = V(g_1 \cdots g_n)$, and let $F : k^m \setminus W \to k^n$, be the function defined by equations (36). Set $g = g_1 \cdots g_n$. Consider the ideal

$$J = \langle f_1 - g_1 x_1, \ldots, f_n - g_n x_n, 1 - gy \rangle \subset k[y, t_1, \ldots, t_m, x_1, \ldots, x_n],$$

and let

$$J_{m+1} = J \cap k[x_1, \ldots, x_n].$$  

(37)

Then $V(J_{m+1})$ is the smallest variety in $k^n$ containing $F(k^m \setminus W)$.

$J_{m+1}$ is computing using the Elimination Theorem.
Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$ and let $G$ be a Groebner basis for an ideal $I$ of $k[x_1, \ldots, x_n]$ with respect to this order. Then for every $\ell$, $0 \leq \ell \leq n - 1$, the set $G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$ is a Groebner basis for the ideal $I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n]$ (the $\ell$–th elimination ideal of $I$).
Theorem (Hu, Han, R., 2013)

The Zariski closure of all time-reversible (with respect to (32)) systems inside the family (30) with coefficients in the field $k$ ($k$ is $\mathbb{R}$ or $\mathbb{C}$) is the variety $\mathbf{V}(I_S)$ of the ideal

$$I_S = k[a, b, c] \cap H. \quad (38)$$

A generating set for $I_S$ (called the Sibirsky ideal) is obtained by computing a Groebner basis for $H$ with respect to any elimination order with $\{y, \gamma\} > \{a, b, c\}$ and choosing from the output list the polynomials which do not depend on $y$ and $\gamma$.

Corollary

Let $I_S$ be ideal (38) of system (20). Then all systems from $\mathbf{V}(I_S)$ are integrable.
\[ \dot{x}_1 = x_1 A_1(x_1, x_2, x_3), \quad \dot{x}_2 = x_2 (1 + A_2(x_1, x_2, x_3)), \quad \dot{x}_3 = -x_3 (1 + A_3(x_1, x_2, x_3)) \] (39)

**Theorem**

Suppose \( A_j(x, y, z) \) is a homogeneous polynomial function of degree \( m \), \( j \in \{1, 2, 3\} \) and that system (39) is transformed to system

\[ \dot{y}_1 = y_1 B_1(y_1, y_2, y_3), \quad \dot{y}_2 = -y_2 (1 + h(y_1, y_2, y_3)),\quad \dot{y}_3 = y_3 (1 - h(y_1, y_2, y_3)) \] (40)

by

\[ y_1 = \frac{k_1 x_1}{f^{1/m}}, \quad y_2 = \frac{k_2 x_3}{f^{1/m}}, \quad y_3 = \frac{k_3 x_2}{f^{1/m}} \] (41)

where \( f = 1 + F \) and \( F(x, y, z) \) is homogeneous polynomial function of degree \( m \).

If \( B(y_1, y_3, y_2) = -B(y_1, y_2, y_3) \) and \( h(y_1, y_3, y_2) = -h(y_1, y_2, y_3) \) then system (39) has two functionally independent local analytic first integrals in a neighborhood of the origin.
\[ \dot{x} = x(a_{200}x + a_{110}y + a_{101}z), \]
\[ \dot{y} = -y + b_{200}x^2 + b_{110}xy + b_{101}xz + b_{020}y^2 + b_{002}z^2, \]
\[ \dot{z} = z + c_{200}x^2 + c_{110}xy + c_{101}xz + c_{020}y^2 + c_{002}z^2. \]
To find the necessary conditions for existence of integrals

\[
\phi = x + \sum_{i+j+k>1} \phi_{ijk}x^i y^j z^k \tag{43}
\]

\[
\psi = yz + \sum_{i+j+k>2} \psi_{ij}x^i y^j z^k \tag{44}
\]

using the computer algebra system Mathematica we computed polynomials \( g_\alpha \) and \( h_\alpha \) defined according to (26) and (27) up to \( |\alpha| \leq 8 \). As the result of the calculations we have obtained the ideal

\[
B_8 = \langle g_\alpha, h_\alpha \mid \alpha \in \mathcal{R}, |\alpha| \leq 8 \rangle.
\]
Then, we tried to find the irreducible decomposition of the variety $\mathbf{V}(B_8)$ of the ideal $B_8$ using the routine $\text{minAssGTZ}$ of the computer algebra system $\text{Singular}$. It was not possible to complete computations on our facilities. However the linear transformation

$$y \mapsto by, \quad z \mapsto cz,$$

where $bc \neq 0$, brings (42) to a quadratic system with the same linear part and $b_{011}$ is changed to $b_{011}/c$, and $c_{011}$ is changed to $c_{011}/b$. Thus, to obtain the necessary conditions for integrability of system (42) it is sufficient to consider separately the following four cases:

(i)$b_{011} = c_{011} = 0$

(ii)$b_{011} = 0c_{011} = 1$

(iii)$b_{011} = 1, c_{011} = 0$

(iv)$b_{011} = c_{011} = 1$. 

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Integrability of Polynomial Systems of ODEs
Consider three dimensional system (42) with $b_{011} = c_{011} = 0$. The system is integrable if and only if $a_{200} = 0$ and one of the following conditions is satisfied:

1) $c_{200} = b_{110} + c_{101} = b_{200} = a_{101} c_{110} + b_{101} c_{020} - c_{110} c_{002} - a_{110} c_{101}$
   \[= a_{110} b_{101} - b_{020} b_{101} + b_{002} c_{110} + a_{101} c_{101} = 0,\]

2) $\begin{align*}
   b_{002}^2 c_{110}^3 + b_{101}^2 c_{020}^2 &= b_{020}^2 b_{101}^2 c_{020} - b_{002}^2 c_{110}^2 c_{002} = b_{020} b_{002} c_{110} + b_{101} c_{020}^2 \\
   &= b_{020} b_{101} + c_{110} c_{002} = b_{020}^2 b_{002} - c_{020} c_{002} = b_{002} c_{110} + b_{200} b_{101} c_{020} = b_{200} b_{002} c_{110} + b_{101} c_{020}^2 \\
   &= b_{200}^2 b_{101} c_{120} + b_{200} c_{110} c_{002} = b_{002} b_{101} c_{120} + b_{200} b_{101} c_{020} = b_{200} b_{002} c_{110} + b_{101} c_{020} \\
   &= b_{200} b_{020} - c_{200} c_{002} = b_{002}^2 c_{200}^2 + b_{200}^2 c_{020} c_{002} = b_{101} c_{200} + b_{200} c_{110} c_{002} \\
   &= b_{002}^3 c_{200}^2 + b_{200}^3 c_{020} = -a_{101} b_{020} + a_{110} c_{002} = a_{110} b_{101} c_{200} + a_{101} b_{200} c_{110} \\
   &= -a_{101} b_{002} c_{110} + a_{110} b_{101} c_{020} = a_{110} b_{002} c_{110} + a_{101} b_{101} c_{020} = a_{110} b_{002} c_{110} \\
   &= a_{110} b_{020} b_{101} + a_{110} c_{110} c_{002} = a_{110} b_{020} b_{002} c_{200} - a_{101} b_{200} c_{020} c_{002} \\
   &= a_{110} b_{020} b_{002} - a_{101} c_{020} c_{002} = a_{110} b_{200} - a_{101} c_{200} = a_{110}^2 b_{101} + a_{101}^2 c_{110} c_{002} \\
   &= a_{110}^2 b_{002} c_{200} - a_{101}^2 b_{200} c_{020} = a_{110}^2 b_{200} b_{002} - a_{101}^2 c_{020} c_{002} = a_{110}^3 b_{002} - a_{101}^3 c_{110} \\
   &= b_{110} + c_{101} = 0,
\end{align*}\]

3) $c_{002} = b_{020} = b_{110} + c_{101} = a_{101} = a_{110} = 0.$
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Thank you for your attention!