Integrability of Polynomial Systems of Ordinary Differential Equations

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Based on the works:

- Z. Hu, M. Han and V.R., Local Integrability of a Family of Three-Dimensional Quadratic Systems, Physica D, **265** (2013) 78-86.

- V. R. Y.-H. Xia and X. Zhang. Varieties of local integrability of analytic differential systems and their applications. J. Differential Equations, **257** (2014) 3079-3101.

- V. R. and D. S. Shafer, Time-reversibility of a 3-dim system, preprint, 2015.

Consider the system

$$\dot{u} = -v + p(u, v), \quad \dot{v} = u + q(u, v), \tag{1}$$

where p and q are convergent series without free and linear terms. It has a center at the origin (all trajectories are ovals) iff it is locally analytically equivalent to a system of the form

$$\dot{x} = ix(1 + g(xy)), \quad \dot{y} = -iy(1 + g(xy)), \tag{2}$$

where, $i = \sqrt{-1}$, $x = u + iv$ and $y = \bar{x}$.
$\Longrightarrow$ xy is a first integral of (2)
$\Longrightarrow$ $u^2 + v^2 + h.o.t.$ is a first integral of (1)

Theorem (Poincaré-Lyapunov)

System (1) has a center at the origin iff it admits a first integral of the form $u^2 + v^2 + h.o.t.$

We discuss a generalization of the center problem (the Poincaré integrability problem) to n-dim systems.

$$\dot{x} = Ax + \mathbf{f}(x), \tag{3}$$

A is $n \times n$ matrix, $x = (x_1, \ldots, x_n)^\tau$, $\mathbf{f}(x) = (f_1(x), \ldots, f_n(x))^\tau$, and f_i are series starting with at least quadratic terms.
Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the n–tuple of eigenvalues of A. Set $\mathbb{Z}_+ = \mathbb{N} \cup 0$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ denote

$$\langle \lambda, \alpha \rangle = \sum_{i=1}^{n} \alpha_i \lambda_i$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let

$$\mathfrak{R} = \{\alpha \in \mathbb{Z}_+^n | \quad \langle \lambda, \alpha \rangle = 0, \ |\alpha| > 0\},$$

and denote by r_λ the rank of vectors in the set $\mathfrak{R}$.

A substitution

$$x = \Phi(y) := y + \varphi(y), \tag{4}$$

transforms (3) to its Poincaré–Dulac normal form, i.e. a system of the form

$$\dot{y} = Ay + \mathbf{g}(y), \tag{5}$$

where $\mathbf{g}(y) = (g_1(y), \ldots, g_n(y))^\tau$ contains only resonant terms, that is, each monomial in g_k, $k = 1, \ldots, n$, is of the form $g^{(\alpha)} y^\alpha e_k$ with

$$\langle \lambda, \alpha \rangle - \lambda_k = 0,$$

where e_k is the n–dimensional unit vector with its nth component equal to 1 and the others all equal to zero. We call the transformation (4) a *normalization*.

The normalization containing only nonresonant terms is unique. We call this normalization a *distinguished normalization* and term the corresponding Poincaré–Dulac normal form a *distinguished normal form*.

Convergence of the normalizing transformation

Normalization (4) does not necessarily converge, so generally speaking φ and $\mathbf{g}$ are formal power series.

Poincaré domain in $\mathbb{C}^n$ is the set of all points $(z_1, \ldots, z_n)$ such that the convex hull of the set $\{z_1, \ldots, z_n\} \subset \mathbb{C}$ does not contain the origin. Then if the vector $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues of A in (3) lies in the Poincaré domain then there exists a convergent normalizing transformation.

Theorem (C. L. Siegel)

Suppose there exist positive constants $C > 0$ and $\nu > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| > 1$ and for all $k \in \{1, \ldots, n\}$ the inequality

$$\left| \sum_{i=1}^{n} \alpha_i \lambda_i - \lambda_k \right| \geq C|\alpha|^{-\nu} \tag{6}$$

holds. Then there exists a convergent transformation of (3) to normal form.

Theorem (V. A. Pliss)

Suppose that for system (3)
(i) the nonzero elements among the $\sum_{j=1}^{n} \alpha_j \lambda_j - \lambda_k$ *satisfy condition* (6)
(ii) some formal normal form of (3) *is linear.*
Then there exists a convergent transformation to normal form.

Bryuno conditions that together are sufficient for existence of a convergent normalizing transformation:

1) *Condition* ω: for $w_\ell = \min(\alpha, \lambda)$ over all $\alpha \in \mathbb{N}_0^n$ for which $(\alpha, \lambda) \neq 0$ and $|\alpha| \leq 2^l$, $\sum 2^{-\ell} \ln w_\ell < \infty$;
2) *Condition A* (simplified version): some normal form has the form

$$\dot{\mathbf{y}} = (1 + g(\mathbf{y}))A\mathbf{y}, \tag{7}$$

that is, $\dot{y}_j = \lambda_j y_j (1 + g(\mathbf{y}))$ for some scalar function $g(\mathbf{y})$.

Following to S. Walcher we say that (3) satisfies the Pliss-Bryuno condition if it can be transformed to (7) by a normalizing transformation.

For simplicity we assume that A is in Jordan normal form and lower triangular.

Definition

System (3) is *(locally) analytically* (or *formally*) *integrable* if it has $n-1$ functionally independent analytic (or formal) first integrals in a neighborhood of the origin.

Theorem (X. Zhang, Llibre-Pantazi-Walcher)

System (3) *has* $n-1$ *functionally independent analytic first integrals in a neighborhood of the origin if and only if* $r_\lambda = n-1$ *and the distinguished normal form of* (3) *satisfies the Pliss-Bruno condition.*

Example: Recursive Construction of a Formal First Integral

$$\mathcal{X}: \quad \begin{aligned} \dot{u} &= \quad -v + P(u,v,w) = \widetilde{P}(u,v,w) \\ \dot{v} &= \qquad u + Q(u,v,w) = \widetilde{Q}(u,v,w) \\ \dot{w} &= -\lambda w + R(u,v,w) = \widetilde{R}(u,v,w) \end{aligned} \qquad \lambda \in \mathbb{R} \setminus \{0\} \tag{8}$$

P, Q, and R are real analytic in a neighborhood of the origin.
We look for a function $\Phi(u,v,w)$ with undetermined coefficients $\phi_{jk\ell}$,

$$\Phi(u,v,w) = u^2 + v^2 + \sum_{j+k+\ell=3} \phi_{jk\ell} u^j v^k w^\ell, \tag{9}$$

such that

$$\frac{\partial \Phi}{\partial u}\widetilde{P} + \frac{\partial \Phi}{\partial v}\widetilde{Q} + \frac{\partial \Phi}{\partial w}\widetilde{R} \equiv 0. \tag{10}$$

Obstacles for the fulfillment of (10) will give us the necessary conditions for the existence of a first integral of the form

$$\Phi(u,v,w) = u^2 + v^2 + \dots. \tag{11}$$

A computational procedure to find the first $m-1$ conditions for integrability is as follows.

- Write down the initial string of (9) up to order $2m$, $\Phi_{2m}(u,v,w) = u^2 + v^2 + \sum_{j+k+\ell=3}^{2m} \phi_{jk\ell} u^j v^k w^\ell$.
- For each $i = 3, \ldots, 2m+1$ equate coefficients of terms of order i in the expression

$$\frac{\partial \Phi_{2m}}{\partial u}\widetilde{P} + \frac{\partial \Phi_{2m}}{\partial v}\widetilde{Q} + \frac{\partial \Phi_{2m}}{\partial w}\widetilde{R} - g_1(u^2+v^2)^2 - \cdots - g_{m-1}(u^2+v^2)^m \quad (12)$$

to zero obtaining $2m-2$ systems of linear variables in unknown variables $\phi_{jk\ell}$.

Computing in this way one obtains a list of polynomials, $g_1, g_2, g_3, \dots$ in parameters of system (8). We call the polynomial g_i the *i-th focus quantity (Lyapunov number)*. Each polynomial g_i represents an obstacle for existing of integral (9), that is, system (8) admits an integral (11) iff

$$g_1 = g_2 = g_3 = \cdots = 0.$$

The set of systems with a first integral of the form (11) is the set of common zeros of an infinite system of polynomials

$$g_1 = g_2 = g_3 = \cdots = 0. \tag{13}$$

Conditions (13) are *the necessary conditions* for existence of first integral $\Phi(u, v, w) = u^2 + v^2 + \dots$ in system (8).

Two difficulties in computing the necessary conditions for integrability:

1) Polynomials $g_1, g_2, g_3, \dots$ are not uniquely defined (depend on the choice of resonant terms).
Let $\mathcal{X}$ be the vector field associated to system (3).
Let $\psi(x)$ be a series. We call the term $\psi^{(\alpha)} x^\alpha$ a resonant term if $\alpha \in \mathfrak{R}$ ($\langle \alpha, \lambda \rangle = 0$).

2) Solving even a finite system of polynomials

$$g_1 = g_2 = g_3 = \cdots = g_k = 0$$

can be an extremely laborious problem.

Uncertain choice of g_i

Theorem (VR, Y. Xia, X. Zhang, J. Differential Equations, 2014)

For system (3) the following statements hold.
(a) There exist series $\psi(x)$ with its resonant monomials arbitrary such that

$$\mathcal{X}(\psi(x)) = \sum_{\alpha \in \mathfrak{R}} p_\alpha x^\alpha, \tag{14}$$

where p_α are functions of the coefficients of (3).
(b) If the vector field (3) has $n-1$ functionally independent analytic or formal first integrals, then for any ψ satisfying (14), we have

$$p_\alpha = 0, \qquad \text{for all} \quad \alpha \in \mathfrak{R}. \tag{15}$$

(c) Assume that the rank of $\mathfrak{R}$ is k, i.e. $r_\lambda = k$, and there are k functionally independent $\psi^{(1)}, \ldots, \psi^{(k)}$, such that for the corresponding coefficients in (14) hold $\quad p_\alpha^{(i)} = 0, \quad$ for all $\alpha \in \mathfrak{R}$, $i = 1, \ldots, k$. Then the vector field $\mathcal{X}$ has exactly k functionally independent analytic or formal first integrals.

Definition

The variety of an ideal I generated by $f_1(x_1, \ldots, x_n), \ldots, f_1(x_1, \ldots, x_n)$ of the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ is the set of all points in $\mathbb{F}^n$ where all polynomials of I vanish. (The variety of I is denoted by $\mathbf{V}(I)$).

W.l.o.g we can take $\psi_\alpha^{(i)} = 0$ for resonant α. Then p_α are polynomials. Denote by $\mathcal{B}$ the ideal generated by the polynomials p_α, for some choice of $n-1$ functionally independent functions $\psi^{(1)}, \ldots, \psi^{(n-1)}$ satisfying (14), i.e.

$$\mathcal{B} = \langle p_\alpha^{(i)} | \ \alpha \in \mathfrak{R}, \quad i = 1, \ldots, n-1 \rangle. \tag{16}$$

By the equivalence of (b) and (c) with $k = n-1$ the variety of $\mathcal{B}$, $\mathbf{V}(\mathcal{B})$, is the set of *all points* in the space of parameters of system (3), such that the corresponding systems have $n-1$ functionally independent integrals. We call $\mathbf{V}(\mathcal{B})$ the *integrability variety* of system (3).

To find the variety of $\mathcal{B}$ we can choose $n-1$ linearly independent vectors from $\mathfrak{R}$, let say $\alpha_1, \dots, \alpha_{n-1} \in \mathfrak{R}$. Then $x^{\alpha_1}, \dots, x^{\alpha_k}$ are functionally independent (integrals of the system of the linear approximation) and we look for $n-1$ functions $\psi_s(x) = x^{\alpha_s} +$ *higher order terms* satisfying

$$\mathcal{X}(\psi^{(s)}(x)) = \sum_{\alpha \in \mathfrak{R}} p_\alpha^{(s)} x^\alpha.$$

In actual calculations we can find only a finite number of polynomials $p_\alpha^{(s)}$, so we compute few first polynomials $p_\alpha^{(s)}$ which generate some ideal $\mathcal{B}_m$. Then,

a) we find the irreducible decomposition of $\mathbf{V}(\mathcal{B}_m)$ (solve the polynomial system $p_\alpha^{(s)} = 0$),

b) using different methods we try to show that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_m)$, that is, all systems corresponding to points from $\mathbf{V}(\mathcal{B}_m)$ have $n-1$ functionally independent analytic or formal first integrals.

Solving polynomial systems

To make a progress it is crucial to have an efficient approach for solving systems of polynomials of many variables:

$$
\begin{aligned}
f_1(x_1, \dots, x_n) &= 0, \\
&\dots\dots\dots\dots\dots\dots \\
f_m(x_1, \dots, x_n) &= 0.
\end{aligned}
\tag{17}
$$

Let us find the variety in $\mathbb{C}^3$ of the ideal $I = \langle f_1, f_2, f_3, f_4 \rangle$, where

$$
\begin{aligned}
f_1 &= 8x^2y^2 + 5xy^3 + 3x^3z + x^2yz, \\
f_2 &= x^5 + 2y^3z^2 + 13y^2z^3 + 5yz^4, \\
f_3 &= 8x^3 + 12y^3 + xz^2 + 3, \\
f_4 &= 7x^2y^4 + 18xy^3z^2 + y^3z^3.
\end{aligned}
\tag{18}
$$

that is, the solution set of the system
$f_1 = 0, \quad f_2 = 0, \quad f_3 = 0, \quad f_4 = 0$. Under the lexicographic ordering with $x > y > z$ a Gröbner basis for I is $G = \{g_1, g_2, g_3\}$, where $g_1 = x$, $g_2 = y^3 + \frac{1}{4}$, $g_3 = z^2$. $f_1 = f_2 = f_3 = f_4 = 0 \iff g_1 = g_2 = g_3 = 0$

This method ALWAYS works when the set of solution is finite: compute a Gröbner basis with respect to a lexicographic order, the basis MUST be triangular (like in Gauss row-echelon form, but with non-linear equations). We have the following computational obstacle:
in the example below the following polynomial appears in the intermediate computations of the Gröbner basis:

$$y^3 - 17359065042904512907647471 82.... \tag{19}$$

The integer in the second term of the above polynomial contains roughly 80,000 digits.

- At least theoretically the Groebner basis theory allows to solve polynomial systems with a finite number of solutions.

Infinite number of solutions

In generic case the variety consists of infinitely many points.

"To solve" a polynomial system means to find a decomposition of the variety of the ideal (the zero set) into irreducible components, that is, to find a representation $V = V_1 \cup \cdots \cup V_m$, where each V_i is irreducible.

Example. For $J = \langle xy, xz \rangle$, the variety of J ($xy = zx = 0$) is the union of the plane $x = 0$ and the line $y = z = 0$.

There are 3 algorithms for irreducible decompositions, all implemented in Singular:
G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2005). `http://www.singular.uni-kl.de`.
– Gianni–Trager–Zacharias (1988) (minAssGTZ)
– Shimoyama–Yokoyama (1996) (primdecSY)
– Characteristic sets method (Wang, 1992) (minAssChar)
(the first one is implemented also in Maple)

```
>LIB "primdec.lib";
>ring r=0,(a20,a11,a02,a13,b31,b20,b11,b02),dp;
>poly g11=a11-b11;
>poly g22=a20*a02-b02*b20;
>poly g33=(3*a20^2*a13+8*a20*a13*b20+3*a02^2*b31
         -8*a02*b02*b31-3*a13*b20^2-3*b02^2*b31)/8;
>poly g44=(-9*a20^2*a13*b11+a11*a13*b20^2
         +9*a11*b02^2*b31-a02^2*b11*b31)/16;
>poly g55=(-9*a20^2*a13*b02*b20+a20*a02*a13*b20^2
         +9*a20*a02*b02^2*b31+18*a20*a13^2*b20*b31
         +6*a02^2*a13*b31^2-a02^2*b02*b20*b31
```

```
>ideal i = g11,g22,g33,g44,g55;
>minAssGTZ(i);
[1]:
       _[1]=a02-3*b02
       _[2]=a11-b11
       _[3]=3*a20-b20
[2]:
       _[1]=b11
       _[2]=3*a02+b02
       _[3]=a11
       _[4]=a20+3*b20
       _[5]=3*a13*b31+4*b20*b02
[3]:
       _[1]=a11-b11
       _[2]=a20*a02-b20*b02
       _[3]=a20*a13*b20-a02*b31*b02
       _[4]=a02^2*b31-a13*b20^2
       _[5]=a20^2*a13-b31*b02^2
```

Modular arithmetic approach

The notorious computational difficulty of the Gröbner basis calculations over the field of rational numbers is an essential obstacle for using the Gröbner basis theory for the real world applications.

Modular calculations: choose a prime number p and do all calculations modulo p, that is, in the finite field of the characteristic p (the field $\mathbb{Z}_p = \mathbb{Z}/p$). The modular calculations still keep essential information on our original system and it is often possible to extract this information from the result of calculations in $\mathbb{Z}_p$ and to obtain the exact solution of polynomial system over the field of rational numbers.

P. Wang's algorithm for the rational reconstruction
Step 1. $u = (u_1, u_2, u_3) := (1, 0, m)$, $v = (v_1, v_2, v_3) := (1, 0, c)$
Step 2. While $\sqrt{m/2} \leq v_3$ do
$\{q := \lfloor u_3/v_3 \rfloor$, $r := u - qv, u := v$, $v := r\}$
Step 3. If $|v_2| \geq \sqrt{m/2}$ then error()
Step 4. Return v_3, v_2
$\lfloor \cdot \rfloor$ stands for the floor function.
Given an integer c and a prime number p the algorithm produces integers v_3 and v_2 such that $v_3/v_2 \equiv c \pmod p$, that is, $v_3 = v_2 c + pt$ with some t. If such a number v_3/v_2 does need not exist the algorithm returns "error()".

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Step 2. While $\sqrt{m/2} \le v_3$ do

$\{q := \lfloor u_3/v_3 \rfloor,\ r := u - qv, u := v,\ v := r\}$

Step 3. If $|v_2| \ge \sqrt{m/2}$ then error()

Step 4. Return v_3, v_2

$\lfloor \cdot \rfloor$ stands for the floor function.

Given an integer c and a prime number p the algorithm produces integers v_3 and v_2 such that $v_3/v_2 \equiv c \pmod p$, that is, $v_3 = v_2 c + pt$ with some t. If such a number v_3/v_2 does need not exist the algorithm returns "error()".

For the discussed example computing the Gröbner basis of (18) over the field of characteristic 32003 we find $G = \{x,\ y^3 + 8001,\ z^2\}$.

Rational reconstruction yields $8001 \equiv 1/4 \pmod{32003}$. Therefore the reconstructed (lifted) Gröbner basis is $G = \{x,\ y^3 + 1/4,\ z^2\}$.

Radical Membership Test

For a polynomial f and an ideal $I = \langle f_1, \ldots, f_m \rangle$ in $k[x_1, \ldots, x_n]$, $k = \mathbb{C}$, f is equal to zero on $\mathbf{V}(I)$) if and only if the reduced Gröbner basis of the ideal $\langle 1 - wf, f_1, \ldots, f_m \rangle$ (here w is a new variable) is equal to $\{1\}$.

Allows to check if zero sets of $I = \langle f_1, \ldots, f_m \rangle$ and $J = \langle h_1, \ldots, h_s \rangle$ are the same in $\mathbb{C}^n$.

Decomposition Algorithm with Modular Arithmetics

(VR and M. Prešern, *J. Comput. Appl. Math.*, 2011)

- Choose a prime number p and compute the minimal associated primes $\tilde{Q}_1, \ldots, \tilde{Q}_s$ of $I = \langle f_1, \ldots, f_s \rangle$ in $\mathbb{Z}_p[x_1, \ldots, x_n]$.
- Using the rational reconstruction algorithm lift the ideals $\tilde{Q}_i$ $(i = 1, \ldots, s)$ to the ideals Q_i in $\mathbb{Q}[x_1, \ldots, x_n]$
- For each i using the radical membership test check whether the original polynomials $f_1, \ldots, f_s$ vanish on the components Q_i of the decomposition (on $\mathbf{V}(Q_i)$), i.e. whether the reduced Gröbner basis of the ideal $\langle 1 - wf, Q_i \rangle$ is equal to $\{1\}$. If "yes", then go to the step 4, otherwise take another prime p and go to step 1.
- Compute $Q = \cap_{i=1}^{s} Q_i \subset \mathbb{Q}[x_1, \ldots, x_n]$.
- Check that $\sqrt{Q} = \sqrt{I}$, i.e, $\forall\ g \in Q$ the reduced GB of the ideal $\langle 1 - wg, I \rangle$ is $\{1\}$ and $\forall\ f \in I$ the reduced GB of $\langle 1 - wf, Q \rangle$ is equal to $\{1\}$. If it is the case then $\mathbf{V}(I) = \cup_{i=1}^{s} \mathbf{V}(Q_i)$. If not, then choose another prime p and go to Step 1.

Example: a 3-dim system

System with $(0:-1:1)$ resonant point at the origin:

$$\begin{aligned}
\dot{x_1} &= \sum_{i+j+k=2}^{m} p_{ijk} x_1^i x_2^j x_3^k = P(x), \\
\dot{x_2} &= -\, x_2 + \sum_{i+j+k=2}^{m} q_{ijk} x_1^i x_2^j x_3^k = Q(x), \\
\dot{x_3} &= \ \ x_3 + \sum_{i+j+k=2}^{m} r_{ijk} x_1^i x_2^j x_3^k = R(x),
\end{aligned} \tag{20}$$

P, Q, R are polynomials and $x = (x_1, x_2, x_3) \in \mathbb{C}^3$.
For system (20) $\lambda = (0, -1, 1)$, thus, the set $\mathfrak{R}_\lambda$ is

$$\mathfrak{R} = \{\alpha \in \mathbb{N}_+^3 \mid \alpha_2 = \alpha_3\}. \tag{21}$$

$$\psi_1 = x_1 + \sum_{|\alpha|>1} \phi^{(\alpha)} x^\alpha \tag{22}$$

$$\psi_2 = x_2 x_3 + \sum_{|\alpha|>2} \psi^{(\alpha)} x^\alpha. \tag{23}$$

There are series $\psi_1(x)$ and $\psi_2(x)$,

$$\psi_1 = x_1 + \sum_{|\alpha|>1} \psi_1^{(\alpha)} x^\alpha \tag{24}$$

$$\psi_2 = x_2 x_3 + \sum_{|\alpha|>2} \psi_2^{(\alpha)} x^\alpha, \tag{25}$$

such that

$$\frac{\partial \psi_1}{\partial x_1} P + \frac{\partial \psi_1}{\partial x_2} Q + \frac{\partial \psi_1}{\partial x_3} R = \sum_{\alpha \in \mathfrak{R}} g_\alpha(a, b, c) x^\alpha \tag{26}$$

and

$$\frac{\partial \psi_2}{\partial x_1} P + \frac{\partial \psi_2}{\partial x_2} Q + \frac{\partial \psi_2}{\partial x_3} R = \sum_{\alpha \in \mathfrak{R}} h_\alpha(a, b, c) x^\alpha, \tag{27}$$

where P, Q, R are the right hand sides of (20) and g_α, h_α $(\alpha \in \mathfrak{R})$ are polynomials in (a, b, c).

Denote by $\mathcal{B}$ the ideal generated by the polynomials g_α and h_α, $\mathcal{B} = \langle g_\alpha, h_\alpha | \alpha \in \mathfrak{R} \rangle$, and by $\mathbf{V}(\mathcal{B})$ its variety – $\mathbf{V}(\mathcal{B})$ is the *integrability variety* of (20).

The set of all integrable systems (20) in the space of parameters of the system is the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B}$ and it is the same for any choice of series (24) and polynomials g_α, h_α ($\alpha \in \mathfrak{R}$) satisfying (26) and (27).

After decomposition of $\mathbf{V}(\mathcal{B})$ using the decomposition algorithm with modular arithmetic we obtain the necessary condition of integrability. The next step: prove their sufficiency.

Two main mechanisms for integrability:

- Darboux integrability
- Time-reversibility

$$\frac{d\mathbf{z}}{dt} = F(\mathbf{z}) \quad (\mathbf{z} \in \Omega), \tag{28}$$

$F : \Omega \mapsto T\Omega$ is a vector field and Ω is a manifold.

Definition

A time-reversible symmetry of (28) is an invertible map $T : \Omega \mapsto \Omega$, such that

$$\frac{d(T\mathbf{z})}{dt} = -F(T\mathbf{z}). \tag{29}$$

By Llibre, Pantazi and Walcher (2012) if a system (30) is time-reversible with respect to a linear invertible transformation which permutes x_2 and x_3 then it is integrable.

$$\dot{x}_1 = \sum a_{jkl} x_1^j x_2^k x_3^l, \quad \dot{x}_2 = x_2 \sum b_{mnp} x_1^m x_2^n x_3^p, \quad \dot{x}_3 = x_3 \sum c_{qrs} x_1^q x_2^r x_3^s. \tag{30}$$

Let u, v, w be the number of parameters of the first, the second and the third equation, respectively. By (a, b, c) we denote the $(u + v + w)$-tuple of parameters of system (30).

System (30) is time-reversible if there exists an invertible matrix T such that

$$T^{-1} \circ f \circ T = -f. \tag{31}$$

We look for a transformation T in the form

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & 1/\gamma & 0 \end{pmatrix}. \tag{32}$$

(31) is satisfied for T defined by (32) if and only if

$$a_{jkl} = -\gamma^{l-k} a_{jlk}, \quad b_{mnp} = -\gamma^{p-n} c_{mpn}. \tag{33}$$

Denote by $k[a, b, c]$ the ring of polynomials in parameters of system (30) with the coefficients in a field k and

$$H = \langle 1 - y\gamma,\ a_{jkl} + \gamma^{l-k} a_{jlk},\ b_{mnp} + \gamma^{p-n} c_{mpn} \rangle, \tag{34}$$

where y is a new variable.

Rational implicitization

Suppose we are given the system of equations

$$x_1 = \frac{f_1(t_1, \dots, t_m)}{g_1(t_1, \dots, t_m)}, \ \dots, \ x_n = \frac{f_n(t_1, \dots, t_m)}{g_n(t_1, \dots, t_m)}, \tag{35}$$

where $f_j, g_j \in k[t_1, \dots, t_m]$ for $j = 1, \dots, n$. Let $W = \mathbf{V}(g_1 \cdots g_n)$.
Equations (35) define

$$F: \ k^m \setminus W \ \to \ k^n$$

by

$$F(t_1, \dots, t_m) = \Big(\frac{f_1(t_1, \dots, t_m)}{g_1(t_1, \dots, t_m)}, \ \dots, \ \frac{f_n(t_1, \dots, t_m)}{g_n(t_1, \dots, t_m)}\Big). \tag{36}$$

The image of $k^m \setminus W$ under F denote by $F(k^m \setminus W)$ is not necessarily an affine variety.

Consequently we look for the smallest affine variety that contains $F(k^m \setminus W)$, i.e, its Zariski closure $\overline{F(k^m \setminus W)}$. The problem of finding $\overline{F(k^m \setminus W)}$ is known as the problem of *rational implicitization* (e.g. Cox et al, 2003).

Rational implicitization theorem

Let k be an infinite field, let $f_1, \dots, f_n$ and $g_1, \dots, g_n$ be elements of $k[t_1, \dots, t_m]$, let $W = \mathbf{V}(g_1 \cdots g_n)$, and let $F : k^m \setminus W \to k^n$, be the function defined by equations (36). Set $g = g_1 \cdots g_n$. Consider the ideal

$$J = \langle f_1 - g_1 x_1, \dots, f_n - g_n x_n, 1 - gy \rangle \subset k[y, t_1, \dots, t_m, x_1, \dots, x_n],$$

and let

$$J_{m+1} = J \cap k[x_1, \dots, x_n]. \tag{37}$$

Then $\mathbf{V}(J_{m+1})$ is the smallest variety in k^n containing $F(k^m \setminus W)$.

J_{m+1} is computing using the Elimination Theorem.

Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$ and let G be a Groebner basis for an ideal I of $k[x_1, \ldots, x_n]$ with respect to this order. Then for every ℓ, $0 \leq \ell \leq n-1$, the set $G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$ is a Groebner basis for the ideal $I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n]$ (the ℓ–th elimination ideal of I).

Computation of $\mathcal{I} = k[a, b] \cap H$

Theorem (Hu, Han, R., 2013)

The Zariski closure of all time-reversible (with respect to (32)) systems inside the family (30) with coefficients in the field k (k is $\mathbb{R}$ or $\mathbb{C}$) is the variety $\mathbf{V}(I_S)$ of the ideal

$$I_S = k[a, b, c] \cap H. \tag{38}$$

A generating set for I_S (called the Sibirsky ideal) is obtained by computing a Groebner basis for H with respect to any elimination order with $\{y, \gamma\} > \{a, b, c\}$ and choosing from the output list the polynomials which do not depend on y and γ.

Corollary

Let I_S be ideal (38) of system (20). Then all systems from $\mathbf{V}(I_S)$ are integrable.

A generalization in the case of Lotka-Volterra system, V.R. & D. Shafer, preprint, 2015

$$\dot{x}_1 = x_1 A_1(x_1, x_2, x_3),\ \dot{x}_2 = x_2(1 + A_2(x_1, x_2, x_3)),\ \dot{x}_3 = -x_3(1 + A_3(x_1, x_2, x_3)) \tag{39}$$

Theorem

Suppose $A_j(x, y, z)$ is a homogeneous polynomial function of degree m, $j \in \{1, 2, 3\}$ and that system (39) is transformed to system

$$\dot{y}_1 = y_1 B_1(y_1, y_2, y_3), \dot{y}_2 = -y_2(1 + h(y_1, y_2, y_3)), \dot{y}_3 = y_3(1 - h(y_1, y_2, y_3)). \tag{40}$$

by

$$y_1 = \frac{k_1 x_1}{f^{1/m}},\ y_2 = \frac{k_2 x_3}{f^{1/m}},\ y_3 = \frac{k_3 x_2}{f^{1/m}}, \tag{41}$$

where $f = 1 + F$ and $F(x, y, z)$ is homogeneous polynomial function of degree m.
If $B(y_1, y_3, y_2) = -B(y_1, y_2, y_3)$ and $h(y_1, y_3, y_2) = -h(y_1, y_2, y_3)$ then system (39) has two functionally independent local analytic first integrals in a neighborhood of the origin.

Example (Hu, Han, R., Physica D, 2013)

$$
\begin{aligned}
\dot{x} &= x(a_{200}x + a_{110}y + a_{101}z),\\
\dot{y} &= -y + b_{200}x^2 + b_{110}xy + b_{101}xz + b_{020}y^2 + b_{002}z^2,\\
\dot{z} &= z + c_{200}x^2 + c_{110}xy + c_{101}xz + c_{020}y^2 + c_{002}z^2.
\end{aligned} \tag{42}
$$

To find the necessary conditions for existence of integrals

$$\phi = x + \sum_{i+j+k>1} \phi_{ijk} x^i y^j z^k \tag{43}$$

$$\psi = yz + \sum_{i+j+k>2} \psi_{ij} x^i y^j z^k \tag{44}$$

using the computer algebra system MATHEMATICA we computed polynomials g_α and h_α defined according to (26) and (27) up to $|\alpha| \leq 8$. As the result of the calculations we have obtained the ideal $B_8 = \langle g_\alpha, h_\alpha | \alpha \in \mathfrak{R}, |\alpha| \leq 8 \rangle$.

Then, we tried to find the irreducible decomposition of the variety $\mathbf{V}(B_8)$ of the ideal B_8 using the routine *minAssGTZ* of the computer algebra system SINGULAR. It was not possible to complete computations on our facilities. However the linear transformation

$$y \mapsto by, \quad z \mapsto cz,$$

where $bc \neq 0$, brings (42) to a quadratic system with the same linear part and b_{011} is changed to b_{011}/c, and c_{011} is changed to c_{011}/b. Thus, to obtain the necessary conditions for integrability of system (42) it is sufficient to consider separately the following four cases:

$$(i) b_{011} = c_{011} = 0 (ii) b_{011} = 0 c_{011} = 1 (iii) b_{011} = 1, c_{011} = 0 (iv) b_{011} = c_{011} = 1.$$

Theorem

Consider three dimensional system (42) with $b_{011} = c_{011} = 0$. The system is integrable if and only if $a_{200} = 0$ and one of the following conditions is satisfied:

1) $$\begin{aligned} & c_{200} = b_{110} + c_{101} = b_{200} = a_{101}c_{110} + b_{101}c_{020} - c_{110}c_{002} - a_{110}c_{101} \\ & = a_{110}b_{101} - b_{020}b_{101} + b_{002}c_{110} + a_{101}c_{101} = 0, \end{aligned}$$

2) $$\begin{aligned} & b_{002}^2 c_{110}^3 + b_{101}^3 c_{020}^2 = b_{020} b_{101}^2 c_{020} - b_{002} c_{110}^2 c_{002} = b_{020} b_{002} c_{110} + b_{101} c_{0} \\ & = b_{020}^2 b_{101} + c_{110} c_{002}^2 = b_{020}^3 b_{002} - c_{020} c_{002}^3 = -b_{020}^2 b_{002} c_{200} + b_{200} c_{020} c_{0}^{2} \\ & = b_{020} b_{101} c_{200} + b_{200} c_{110} c_{002} = b_{002} c_{200} c_{110} + b_{200} b_{101} c_{020} = b_{200} b_{002} c_{11}^{2} \\ & = b_{200} b_{020} - c_{200} c_{002} = -b_{020} b_{002} c_{200}^2 + b_{200}^2 c_{020} c_{002} = b_{101} c_{200}^2 + b_{200}^2 c_{1} \\ & = -b_{002} c_{200}^3 + b_{200}^3 c_{020} = -a_{101} b_{020} + a_{110} c_{002} = a_{110} b_{101} c_{200} + a_{101} b_{200} \\ & = -a_{101} b_{002} c_{110}^2 + a_{110} b_{101}^2 c_{020} = a_{110} b_{002} c_{110} + a_{101} b_{101} c_{020} = a_{110} b_{002} c \\ & = a_{110} b_{020} b_{101} + a_{101} c_{110} c_{002} = a_{110} b_{020} b_{002} c_{200} - a_{101} b_{200} c_{020} c_{002} \\ & = a_{110} b_{020}^2 b_{002} - a_{101} c_{020} c_{002}^2 = a_{110} b_{200} - a_{101} c_{200} = a_{110}^2 b_{101} + a_{101}^2 c_{110} \\ & = a_{110}^2 b_{002} c_{200} - a_{101}^2 b_{200} c_{020} = a_{110}^2 b_{020} b_{002} - a_{101}^2 c_{020} c_{002} = a_{110}^3 b_{002} - \\ & = b_{110} + c_{101} = 0, \end{aligned}$$

3) $$c_{002} = b_{020} = b_{110} + c_{101} = a_{101} = a_{110} = 0.$$

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Thank you for your attention!