APPLICATION OF THE GALI METHOD TO LOCALIZATION DYNAMICS
IN NONLINEAR SYSTEMS

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Abstract
We investigate localization phenomena and stability properties of quasiperiodic oscillations in N degree of freedom Hamiltonian systems and N coupled symplectic maps. In particular, we study an example of a parametrically driven Hamiltonian lattice with only quartic coupling terms and a system of N coupled standard maps. We explore their dynamics using the Generalized Alignment Index (GALI), which constitutes a recently developed numerical method for detecting chaotic orbits in many dimensions, estimating the dimensionality of quasiperiodic tori and predicting slow diffusion in a way that is faster and more reliable than many other approaches known to date.

1. Introduction
Consider a N degree of freedom Hamiltonian system in R2N, described by a Hamiltonian function. The corresponding variational equations of this system along a given trajectory (q(t), p(t)) in R2N are:
\[ \frac{d}{dt} q(t) = J \cdot M(q(t), p(t))\dot{q}(t) \]
where M is the Hessian matrix of the Hamiltonian function. We choose k linearly independent vectors \( \pi_i, i=k+1, \ldots, k \) of unit magnitude for initial conditions. The solutions of the system (1) are k vectors \( \pi(t) = \pi(0) e^{i\beta_{\pi}t} \)
Given k normalized deviation vectors:
\[ \tilde{\omega}(t) = \frac{\pi(t)}{||\pi(t)||} \]
the Generalized Alignment Index of order k has been defined [4] as the volume of a k-dimensional parallelepiped generated by these vectors:
\[ \text{GALI}_k(t) = \left| \text{det}(\tilde{\omega}(t)) \right| \]
The asymptotic behavior of GALI_k is:
\[ \text{GALI}_k(t) \sim e^{\sigma_k t} \]
where \( \sigma_k \) are the Lyapunov exponents and:
\[ \text{GALI}_k(t) = \left| \text{det}(\tilde{\omega}(t)) \right| \]
if the motion is regular and lies on a torus of dimensionality \( m \leq N \) lower than N [3,5].

2. Applications to the dynamics near discrete breathers
We start with initial conditions near the exact breather and check whether the motion remains quasiperiodic or becomes chaotic. We accomplish this by computing the GALI indices along the reference orbit. If the breather is stable, the GALI method can be used to determine the dimensionality of the torus surrounding the breather in the 2N dimensional phase space. As for chaotic motion diffusing slowly away from these breathers, it is rapidly and efficiently predicted by the exponential convergence of all GALI indices to zero.
We choose a parametrically driven Hamiltonian lattice of anharmonic oscillators [2], which involves only quartic coupling terms and hence presents strong localization phenomena due to the absence of phonons:
\[ H(0) = \sum_j \frac{1}{2} \dot{\varphi}_j^2 + \frac{1}{2} \left( 1 - \cos(\alpha_0 \varphi_j) \right) \dot{\varphi}_j^2 - \frac{1}{4} \varphi_j^4 - \frac{K}{4} \varphi_j^4 \]
where \( \alpha_0 \) and \( \varphi_j \) are the frequency and amplitude of the driver respectively. It has been shown [1] that the periodic orbits obeying Hamiltonian (6) can be written as a product of k normal modes of the model and we choose initial conditions that excite only a small number of them. Thus, we extract the small non-zero value implying that motion lies on a 5D torus. (Fig. 3b) Time evolution of GALI indices showing the orbits lie on a 5D torus.

3. Application to the dynamics of N coupled standard maps
Let us consider a system of N coupled standard maps described by the following equations:
\[ x_{n+1}^j = \alpha_{n+1}^j + \beta_{n+1}^j \]
\[ y_{n+1}^j = y_{n}^j + \beta_{n+1}^j \sin(2\pi x_{n}^j) \]
with \( j = 1, \ldots, N, \) fixed boundary conditions \( x_0 = x_{2N} = 0 \) and \( \beta \) the coupling constant.
Taking \( N=20 \) and \( \beta = 10^3, \) we look for localized oscillations taking as initial conditions \( (x,y) = (0.5,0) \), first perturb only the 11th particle and then perturb the 11th and the 12th particle.

Fig. 2. (a) GALI indices for: (a) D=1.1 and initial energy \( H(0)=0.133, \) (b) \( \beta=0.9 \) and \( H(0)=0.07. \)
We now perturb slightly the initial conditions of the discrete map \( \Phi(t), \) (Fig. 1b) and observe that the motion now occurs on a 3D torus.

Fig. 3. For \( 1 \leq D \leq 1.2 \), the chaotic character of the orbit is revealed by the exponential decay of all GALI indices.

Fig. 4. (a) GALI_k = 2,...,7 initial conditions R1: (x_1,y_1) = (0.65, 0). GALI_k fluctuates around a non-zero value implying a regular motion that lies on a 2D torus. (b) GALI_k = 2,...,7, initial conditions R2: (x_1,y_1) = (0.65, 0) and (x_2,y_2) = (0.55, 0). Regular motion that lies on a 3D torus.

Fig. 5. (a) Oscillations of particles 16, 17, 18 of the lattice show the localization and quasiperiodic properties of the motion. Here \( D=0.9 \) and \( H(0)=-0.09. \) (b) Time evolution of GALI indices showing the orbits lie on a 3-dimensional torus.

Fig. 6. Time evolution of GALI indices for: (a) D=1.1 and initial energy \( H(0)=0.133, \) (b) D=0.9 and \( H(0)=0.07. \)

6. References