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DYNAMICS OF EPIDEMIC MODELS WITH ASYMPTOMATIC INFECTION AND SEASONAL SUCCESSION

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Abstract. In this paper, we consider a compartmental SIRS epidemic model with asymptomatic infection and seasonal succession, which is a periodic discontinuous differential system. The basic reproduction number $\mathcal{R}_0$ is defined and evaluated directly for this model, and uniform persistence of the disease and threshold dynamics are obtained. Specially, global dynamics of the model without seasonal force are studied. It is shown that the model has only a disease-free equilibrium which is globally stable if $\mathcal{R}_0 \leq 1$, and as $\mathcal{R}_0 > 1$ the disease-free equilibrium is unstable and there is an endemic equilibrium, which is globally stable if the recovering rates of asymptomatic infectives and symptomatic infectives are close. These theoretical results provide an intuitive basis for understanding that the asymptomatically infective individuals and the seasonal disease transmission promote the evolution of the epidemic, which allow us to predict the outcomes of control strategies during the course of the epidemic.

1. **Introduction.** Since Kermack and McKendrick [13] proposed the classical deterministic compartmental model (called SIR model) to describe epidemic outbreaks and spread, mathematical models have become important tools in analyzing the spread and control of infectious diseases, see [1, 2, 5, 9, 11, 12, 20, 21, 27] and references therein. The number of infected individuals used in these models is usually calculated via data in the hospitals. However, some studies on influenza show that some individuals of the population who are infected never develop symptoms, i.e. being asymptomatically infective. The asymptomatically infected individuals will not go to hospital but they can infect the susceptible by contact, then progress to the recovered stage, see for instance [3, 14, 22]. Hence, using the data from hospitals to mathematical models to assess the epidemic will underestimate infection risks.

On the other hand, seasonality is very common in ecological and human social systems (cf. [26]). For example, variation patterns in climate are repeated every

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year, birds migrate according to the variation of season, opening and closing of schools are almost periodic, and so on. These seasonal factors significantly influence the survival of pathogens in the environment, host behavior, and abundance of vectors and non-human hosts. A number of papers have suggested that seasonality plays an important role in epidemic outbreaks and the evolution of disease transmissions, see [4, 6, 8, 9, 16, 17, 19, 21, 28]. However, it is still challenging to understand the mechanisms of seasonality and their impacts on the dynamics of infectious diseases.

Motivated by the above studies on asymptomatic infectivity or seasonality, we develop a compartmental model with asymptomatic infectivity and seasonal factors in this paper. This model is a periodic discontinuous differential system. We try to establish the theoretical analysis on the periodic discontinuous differential systems and study the dynamics of the model. This will allow us to draw both qualitative and quantitative conclusions on the effect of asymptomatic infectivity and seasonality on the epidemic.

The rest of the paper is organized as follows. In section 2, we formulate the SIRS model with asymptomatic infective and seasonal factors, then discuss the existence and regularity of non-negative solutions for this model. In section 3, we define the basic reproduction number $\mathcal{R}_0$ for the model, give the evaluation of $\mathcal{R}_0$ and investigate the threshold dynamics of the model (or the uniformly persistent of the disease). It is shown that the length of the season, the transmission rate and the existence of asymptomatic infection affect the basic reproduction number $\mathcal{R}_0$. In section 4, we study the global dynamics of the model ignoring seasonal factor. We prove that there is a unique disease-free equilibrium and the disease always dies out when $\mathcal{R}_0 \leq 1$; while when $\mathcal{R}_0 > 1$ there is an endemic equilibrium which is global stable if the recovering rates of asymptomatic infective and symptomatic infective are close. A brief discussion is given in the last section.

2. **Model formulation.**

In this section, we first extend the classic SIRS model to a model which incorporates with the asymptomatic infective and seasonal features of epidemics, and then study the regularity of solutions of the model.

Because there are asymptomatically infectious and symptomatically infectious individuals in the evolution of epidemic, the whole population is divided into four compartments: susceptible, asymptomatically infectious, symptomatically infectious and recovered individuals. More precisely, we let S, I_a, I_s and R denote the numbers of individuals in the susceptible, asymptomatic, symptomatic and recovered compartments, respectively, and N be the total population size. Let $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{Z}_+$ be the set of all nonnegative integers, and $\omega > 0$ be given as the period of the disease transmissions. In addition to the assumptions of the classical SIRS model, we list the following assumptions on seasonal factors, asymptomatic infectivity and symptomatic infectivity.

(A1) Due to the opening and closing of schools or migration of birds, each period of the disease transmission is simply divided into two seasons with high and low transmission rates, which are called high season J_2 and low season J_1, respectively. The seasonality is described by a piecewise constant function with high transmission rate β_2 in J_2 and low transmission rate β_1 in J_1, respectively, where $J_1 = [m\omega, m\omega + (1-\theta)\omega)$ and $J_2 = [m\omega + (1-\theta)\omega, (m+1)\omega)$. Here $m \in \mathbb{Z}_+$, and $0 < \theta < 1$ which measures the fraction of the high season to the whole infection cycle.

(A2) There are two classes of infective individuals: asymptomatically infective ones and symptomatically infective ones. Both of them are able to infect susceptible individuals by contact. A fraction μ of infective individuals proceeds to the asymptomatically infective compartment and the remainder (i.e. a fraction $1-\mu$ of infective individuals) goes directly to the symptomatically infective compartment. And the asymptomatically infective and symptomatically infective individuals recover from disease at rate r_a and r_s, respectively.

(A3) The symptomatically infective individuals will get treatment in hospital or be quarantined. Hence, the symptomatic infective individuals reduce their contact rate by a fraction α.

Based on these assumptions, the classical SIRS model can be extended to the following system

$$\begin{cases} \dot{S}(t) = dN(t) - dS(t) - \beta(t)S(t)(I_a(t) + \alpha I_s(t)) + \sigma R(t), \\ \dot{I}_a(t) = \mu\beta(t)S(t)(I_a(t) + \alpha I_s(t)) - (d + r_a)I_a(t), \\ \dot{I}_s(t) = (1-\mu)\beta(t)S(t)(I_a(t) + \alpha I_s(t)) - (d + r_s)I_s(t), \\ \dot{R}(t) = r_a I_a(t) + r_s I_s(t) - (d + \sigma)R(t), \end{cases} \tag{2.1}$$

where $N(t) = S(t) + I_a(t) + I_s(t) + R(t)$, all parameters d, α, σ, μ, r_a and r_s are nonnegative, and

$$\beta(t) = \begin{cases} \beta_1, & t \in J_1 = [m\omega, m\omega + (1-\theta)\omega), \\ \beta_2, & t \in J_2 = [m\omega + (1-\theta)\omega, (m+1)\omega). \end{cases}$$

Parameters β_2 and β_1 are the rates of contact transmission of the disease in high season and low season respectively for which $\beta_2 \geq \beta_1 \geq 0$. Besides, d is both the birth rate and death rate, α, $0 \leq \alpha \leq 1$, is the fraction of the symptomatically infective individuals reducing their contact rate with susceptible, the fraction of infective individuals becoming asymptomatically infective μ satisfies $0 \leq \mu \leq 1$, parameter σ is the rate of recovered population loss of the immunity and reentering the susceptible group, and r_a and r_s are the rates of asymptomatic infective and symptomatic infective recovering with immunity, respectively. Note that the transmission rate $\beta(t)$ is a piecewise function or a step function. This idea has appeared in some literatures such as [19, 24, 15, 10].

From the biological point of view, we focus on the solutions of system (2.1) with initial conditions

$$S(0) = S_0 \geq 0, I_a(0) = I_{a0} \geq 0, I_s(0) = I_{s0} \geq 0, R(0) = R_0 \geq 0 \tag{2.2}$$

in the first octant $\mathbb{R}^4_+$.

Note that

$$\dot{N}(t) = \dot{S}(t) + \dot{I}_a(t) + \dot{I}_s(t) + \dot{R}(t) \equiv 0, \ t \in J_1 \text{ or } \mathrm{t} \in \mathrm{J}_2.$$

Hence, $N(t) = S_0 + I_{a0} + I_{s0} + R_0$, which is a constant for almost all $t \in \mathbb{R}_+$. Since the total population does not change by the assumption, we let

$$S(t) + I_a(t) + I_s(t) + R(t) \equiv N$$

for almost all $t \in \mathbb{R}_+$. Therefore, system (2.1) with the initial conditions (2.2) in $\mathbb{R}^4_+$ can be reduced to

$$\begin{cases} \dot{S} = (d+\sigma)(N-S) - \beta(t)S(I_a + \alpha I_s) - \sigma(I_a + I_s), \\ \dot{I}_a = \mu\beta(t)S(I_a + \alpha I_s) - (d+r_a)I_a, \\ \dot{I}_s = (1-\mu)\beta(t)S(I_a + \alpha I_s) - (d+r_s)I_s, \\ S(0) = S_0, I_a(0) = I_{a0}, I_s(0) = I_{s0}, \\ P_0 = (S_0, I_{a0}, I_{s0}) \in \mathcal{D}_0, \end{cases} \tag{2.3}$$

where $\mathcal{D}_0 \subset \mathbb{R}^3_+$ and

$$\mathcal{D}_0 := \{(S, I_a, I_s) |\ S \geq 0, I_a \geq 0, I_s \geq 0,\ 0 \leq S + I_a + I_s \leq N\}. \tag{2.4}$$

Clearly, the right hand side of system (2.3) is not continuous on the domain $\mathbb{R}_+ \times \mathcal{D}_0$. We claim that the solution of system (2.3) exists globally on the interval $\mathbb{R}_+ = [0, +\infty)$ and is unique.

Theorem 2.1. *For any $P_0 \in \mathcal{D}_0$, system (2.3) has a unique global solution $\varphi(t, P_0) = (S(t, P_0), I_a(t, P_0), I_s(t, P_0))$ in $\mathbb{R}_+$, which is continuous with respect to t and all parameters of this system.*

Moreover, $\varphi(t, P_0) \subseteq \mathcal{D}_0$ for any $t \in \mathbb{R}_+$ and the solution $\varphi(t, P_0)$ is differentiable with respect to P_0, where some derivatives are one-sided if P_0 is on the domain boundary.

Proof. Assume that $\varphi(t, P_0)$ is a solution of system (2.3). We first consider the two systems

$$\begin{cases} \dot{S} = (d+\sigma)(N-S) - \beta_i S(I_a + \alpha I_s) - \sigma(I_a + I_s), \\ \dot{I}_a = \mu\beta_i S(I_a + \alpha I_s) - (d+r_a)I_a, \\ \dot{I}_s = (1-\mu)\beta_i S(I_a + \alpha I_s) - (d+r_s)I_s, \\ S(t_*) = S_*,\ I_a(t_*) = I_{a*},\ I_s(t_*) = I_{s*}, \\ P_* = (S_*, I_{a*}, I_{s*}) \in \mathbb{R}^3_+ \end{cases} \tag{2.5}$$

in the domain $\mathbb{R}_+ \times \mathbb{R}^3_+$, $i = 1, 2$, respectively.

It is clear that for each i the solution of system (2.5) exists and is unique on its maximal interval of existence, and the solution of system (2.5) is differentiable with respect to the initial value P_* by the fundamental theory of ordinary differential equations.

Note that the bounded closed set $\mathcal{D}_0$ in $\mathbb{R}^3_+$ is a positive compact invariant set of system (2.5) since the vector field of system (2.5) on the boundary $\partial\mathcal{D}_0$ of $\mathcal{D}_0$ is directed toward to the interior of $\mathcal{D}_0$ or lies on $\partial\mathcal{D}_0$, where

$$\begin{aligned} \partial\mathcal{D}_0 =& \{(S, I_a, I_s) :\ (S, I_a, I_s) \in \mathbb{R}^3_+, S = 0,\ 0 \leq I_a + I_s \leq N\} \\ & \cup \{(S, I_a, I_s) :\ (S, I_a, I_s) \in \mathbb{R}^3_+, I_s = 0,\ 0 \leq S + I_a \leq N\} \\ & \cup \{(S, I_a, I_s) :\ (S, I_a, I_s) \in \mathbb{R}^3_+, I_a = 0,\ 0 \leq S + I_s \leq N\} \\ & \cup \{(S, I_a, I_s) :\ (S, I_a, I_s) \in \mathbb{R}^3_+, S + I_s + I_a = N\}. \end{aligned}$$

Therefore, the solution of system (2.5) exists globally for any $P_* \in \mathcal{D}_0 \subset \mathbb{R}^3_+$, and these solutions are in $\mathcal{D}_0$ for all $t > 0$.

Let $\phi_i(t, t_*, P_*)$ for $i = 1, 2$ be the solution semiflow of the following system

$$\begin{cases} \dot{S} = (d+\sigma)(N-S) - \beta_i S(I_a + \alpha I_s) - \sigma(I_a + I_s), \\ \dot{I}_a = \mu\beta_i S(I_a + \alpha I_s) - (d+r_a)I_a, \\ \dot{I}_s = (1-\mu)\beta_i S(I_a + \alpha I_s) - (d+r_s)I_s, \\ \phi_i(t_*, t_*, P_*) = P_*,\ P_* \in \mathcal{D}_0, \end{cases} \tag{2.6}$$

respectively, that is, $\phi_i(t,t_*,P_*)=(S(t,t_*,P_*),I_a(t,t_*,P_*),I_s(t,t_*,P_*))$ for $t\geq t_*$ is the solution of system (2.6) with the initial condition $\phi_i(t_*,t_*,P_*)=(S_*,I_{a*},I_{s*})\in\mathcal{D}_0$, respectively.

It follows that the solution $\varphi(t,P_0)$ for $t\geq 0$ of system (2.3) can be determined uniquely by induction. For simplicity, we let $s_m=(m-1)\omega$ and $t_m=s_m+(1-\theta)\omega$ for $m\in\mathbb{Z}_+$. Hence,

$$[0,\infty)=\bigcup_{m=1}^{\infty}[s_m,s_{m+1}]=\bigcup_{m=1}^{\infty}([s_m,t_m]\cup[t_m,s_{m+1}]),$$

and $\varphi(t,P_0)$ can be written as follows.

$$\varphi(t,P_0)=\begin{cases}\phi_1(t,s_1,P_0) & \text{when } t\in[s_1,t_1],\\ \phi_2(t,t_1,\phi_1(t_1,s_1,P_0)) & \text{when } t\in[t_1,s_2],\\ \cdots & \\ \phi_1(t,s_m,u_m) & \text{when } t\in[s_m,t_m],\\ \phi_2(t,t_m,v_m) & \text{when } t\in[t_m,s_{m+1}],\end{cases}\tag{2.7}$$

where u_m and v_m are determined by letting $u_1=P_0$, $v_1=\phi_1(t_1,s_1,u_1)$ and

$$u_m=\phi_2(s_m,t_{m-1},v_{m-1}),\ v_m=\phi_1(t_m,s_m,u_m)\ \text{ for }\ m\geq 2.$$

This implies that the solution $\varphi(t,P_0)$ of system (2.3) exits globally in $\mathbb{R}_+$ and is unique for any $P_0\in\mathcal{D}_0$, and it is continuous with respect to t and all parameters.

By the expression (2.7), it is easy to see that the solution $\varphi(t,P_0)$ lies in $\mathcal{D}_0$ for all $t\geq 0$ and $\varphi(t,P_0)$ is differentiable with respect to P_0. The proof is completed. □

Theorem 2.1 tells us that system (2.3) is ω-periodic with respect to t in $\mathbb{R}_+\times\mathcal{D}_0$, and it suffices to investigate the dynamics of its associated period map $\mathcal{P}$ on $\mathcal{D}_0$ for the dynamics of system (2.3), where

$$\begin{aligned}&\mathcal{P}:\mathcal{D}_0\to\mathcal{D}_0,\\ &\mathcal{P}(P_0)=\varphi(\omega,P_0)=\phi_2(\omega,(1-\theta)\omega,\phi_1((1-\theta)\omega,0,P_0)),\end{aligned}\tag{2.8}$$

which is continuous in $\mathcal{D}_0$.

3. Basic reproduction number and threshold dynamics.

In epidemiology, the basic reproduction number (or basic reproduction ratio) $\mathcal{R}_0$ is an important quantity, defined as the average number of secondary infections produced when an infected individual is introduced into a host population where everyone is susceptible. It is often considered as the threshold quantity that determines whether an infection can invade a new host population and persist. Detailedly speaking, if $\mathcal{R}_0<1$, the disease dies out and the disease cannot invade the population; but if $\mathcal{R}_0>1$, then the disease is established in the population. There have been some successful approaches for the calculations of basic reproduction number for different epidemic models. For example, Diekmann *et al* [7] and van den Driessche and Watmough [23] presented general approaches of calculating $\mathcal{R}_0$ for autonomous continuous epidemic models. And for periodic continuous epidemic models, Wang and Zhao in [25] defined the basic reproduction number. To our knowledge, there is no theoretic approach to calculate the basic reproduction number for periodic discontinuous epidemic models such as system (2.3). In this section, we use the idea and some notations given in [25] to define and calculate the basic reproduction

numbers for system (2.3), and discuss the uniform persistence of the disease and threshold dynamics.

We define $\mathbf{X}$ to be the set of all disease-free states of system (2.3), that is

$$\mathbf{X} = \{(S, I_a, I_s) : \ 0 \le S \le N, I_a = I_s = 0\}.$$

Clearly, the disease-free subspace $\mathbf{X}$ is positively invariant for system (2.3). It can be checked that the period map $\mathcal{P}(P_0)$ in $\mathbf{X}$ has a unique fixed point at $(N, 0, 0)$, which is a unique disease-free equilibrium $(N, 0, 0)$ of system (2.3), denoted by E_0. We now consider a population near the disease-free equilibrium E_0.

For simplicity, we let $\mathbf{x} = (S, I_a, I_s)^T$, and for $i = 1, 2$ set

$$\begin{aligned}
\mathbf{F_i} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu\beta_i N & \alpha\mu\beta_i N \\ 0 & (1-\mu)\beta_i N & \alpha(1-\mu)\beta_i N \end{pmatrix} := \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & F_i \end{pmatrix}, \\
\mathbf{V_i} &= \begin{pmatrix} d+\sigma & \beta_i N + \sigma & \alpha\beta_i N + \sigma \\ 0 & d + r_a & 0 \\ 0 & 0 & d + r_s \end{pmatrix} := \begin{pmatrix} d+\sigma & \mathbf{b_i} \\ \mathbf{0} & V \end{pmatrix}.
\end{aligned}$$

Then the linearized system of (2.3) at E_0 can be rewritten as

$$\frac{d\mathbf{x}}{dt} = (\mathbf{F}(t) - \mathbf{V}(t))\mathbf{x}, \tag{3.1}$$

where $\mathbf{F}(t) = \chi_{J_1}(t)\mathbf{F}_1 \ + \chi_{J_2}(t)\mathbf{F}_2$, $\mathbf{V}(t) = \chi_{J_1}(t)\mathbf{V}_1 + \chi_{J_2}(t)\mathbf{V}_2$, and

$$\chi_{J_i}(t) = \begin{cases} 1 & \text{as } t \in J_i, \\ 0 & \text{as } t \notin J_i. \end{cases}$$

System (3.1) is a piecewise continuous periodic linear system with period ω in $t \in \mathbb{R}_+$. In order to determine the fate of a small number of infective individuals introduced into a disease-free population, we first extend system (3.1) from $t \in \mathbb{R}_+$ to $t \in \mathbb{R}$, and introduce some new notations. When $t \in \cup_{m=-\infty}^{+\infty}(J_1 \cup J_2) = (-\infty, +\infty)$, we set $\mathbb{I}(t) = (I_a(t), I_s(t))^T$, and

$$\mathbb{F}(t) = \chi_{J_1}(t)F_1 \ + \chi_{J_2}(t)F_2 = \begin{pmatrix} \mu N\beta(t) & \alpha\mu N\beta(t) \\ (1-\mu)N\beta(t) & \alpha(1-\mu)N\beta(t) \end{pmatrix},$$

where

$$\beta(t) = \begin{cases} \beta_1, & t \in J_1 = [m\omega, m\omega + (1-\theta)\omega), \\ \beta_2, & t \in J_2 = [m\omega + (1-\theta)\omega, (m+1)\omega), \ m \in \mathbb{Z}. \end{cases}$$

Clearly, $\mathbb{F}(t)$ is a 2×2 piecewise continuous periodic matrix with period ω in $\mathbb{R}$, and it is non-negative. And

$$-V = \begin{pmatrix} -(d+r_a) & 0 \\ 0 & -(d+r_s) \end{pmatrix},$$

which is cooperative in the sense that the off-diagonal elements of $-V$ are non-negative.

Let $Y(t, s)$, $t \ge s$, be the evolution operator of the linear system

$$\frac{d\mathbb{I}(t)}{dt} = -V\mathbb{I}(t). \tag{3.2}$$

Since V is a constant matrix, for each $s \in \mathbb{R}$ the matrix $Y(t, s)$ satisfies

$$\frac{d}{dt}Y(t, s) = -VY(t, s), \ t \ge s, \ Y(s, s) = E^2, \tag{3.3}$$

where E^2 is a 2×2 identity matrix, and $Y(t,s)=e^{-V(t-s)}$. Hence, the monodromy matrix $\Phi_{-V}(t)$ of system (3.2) is $Y(t,0)$, that is,

$$\Phi_{-V}(t)=e^{-Vt}=\begin{pmatrix} e^{-(d+r_a)t} & 0\\ 0 & e^{-(d+r_s)t}\end{pmatrix},$$

where d, r_a and r_s are positive numbers.

We denote $\|\cdot\|_1$ the 1-norm of vectors and matrices. Thus, there exist $K>0$ and $\kappa>0$ such that

$$\|Y(t,s)\|_1\le Ke^{-\kappa(t-s)},\ \forall t\ge s,\ s\in\mathbb{R}.$$

From the boundedness of $\mathbb{F}(t)$, i.e. $\|\mathbb{F}(t)\|_1<K_1$, it follows that there exists a constant $K_1>0$ such that

$$\|Y(t,t-a)\mathbb{F}(t-a)\|_1\le KK_1e^{-\kappa a},\ \forall t\in\mathbb{R},\ a\in[0,+\infty). \tag{3.4}$$

We now consider the distribution of infected individuals in the periodic environment. Assume that $\mathbb{I}(s)$ is the initial distribution of infected individuals in infectious compartments. Then $\mathbb{F}(s)\mathbb{I}(s)$ is the distribution of new infections produced by the infected individuals who were introduced at time s. Given $t\ge s$, then $Y(t,s)\mathbb{F}(s)\mathbb{I}(s)$ is the distribution of those infected individuals who were newly infected at time s and still remain in the infected compartments at time t. Thus, the integration of this distribution from $-\infty$ to t

$$\int_{-\infty}^{t}Y(t,s)\mathbb{F}(s)\mathbb{I}(s)ds=\int_0^{\infty}Y(t,t-a)\mathbb{F}(t-a)\mathbb{I}(t-a)da$$

gives the distribution of cumulative new infections at time t produced by all those infected individuals introduced at times earlier than t.

Let $\mathbb{C}_\omega=\mathbb{C}(\mathbb{R},\mathbb{R}^2)$ be the ordered Banach space of ω-periodic continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$, which is equipped with the norm $\|\cdot\|_c$,

$$\|\mathbb{I}(s)\|_c=\max_{s\in[0,\omega]}\|\mathbb{I}(s)\|_1,$$

and the generating positive cone

$$\mathbb{C}_\omega^+=\{\mathbb{I}(s)\in\mathbb{C}_\omega:\ \mathbb{I}(s)\ge 0,\ s\in\mathbb{R}\}.$$

Define a linear operator $\mathcal{L}:\ \mathbb{C}_\omega\to\mathbb{C}_\omega$ by

$$(\mathcal{L}\mathbb{I})(t)=\int_{-\infty}^{t}Y(t,s)\mathbb{F}(s)\mathbb{I}(s)ds=\int_0^{\infty}Y(t,t-a)\mathbb{F}(t-a)\mathbb{I}(t-a)da. \tag{3.5}$$

It can be checked that the linear operator $\mathcal{L}$ is well defined.

Lemma 3.1. *The operator $\mathcal{L}$ is positive, continuous and compact on $\mathbb{C}_\omega$.*

Proof. Since $Y(t,s)=e^{-V(t-s)}$ and $\mathbb{F}(t)$ is a nonnegative bounded matrix, we get that $\mathcal{L}(\mathbb{C}_\omega^+)\subset\mathbb{C}_\omega^+$. This implies that the linear operator $\mathcal{L}$ is positive.

We now prove the continuity of $\mathcal{L}$. For each $t\in\mathbb{R}$, we have

$$\begin{aligned}\|\mathcal{L}\mathbb{I}(t)\|_1&=\left\|\int_0^{\infty}Y(t,t-a)\mathbb{F}(t-a)\mathbb{I}(t-a)da\right\|_1\\&=\left\|\sum_{j=0}^{\infty}\int_{j\omega}^{(j+1)\omega}Y(t,t-a)\mathbb{F}(t-a)\mathbb{I}(t-a)da\right\|_1\end{aligned}$$

$$\leq \sum_{j=0}^{\infty} \int_{j\omega}^{(j+1)\omega} \|Y(t,t-a)\mathbb{F}(t-a)\mathbb{I}(t-a)\|_1 da$$
$$\leq \sum_{j=0}^{\infty} \int_{j\omega}^{(j+1)\omega} KK_1 e^{-\kappa a}\|\mathbb{I}(t-a)\|_1 da$$
$$\leq \omega K K_1 \sum_{j=0}^{\infty} e^{-\kappa\omega j}\cdot \|\mathbb{I}\|_c$$

by (3.4). Hence,

$$\|\mathcal{L}\mathbb{I}(t)\|_c = \max_{t\in[0,\omega]} \|\mathcal{L}\mathbb{I}(t)\|_1 \leq \omega K K_1 \sum_{j=0}^{\infty} e^{-\kappa\omega j}\cdot \|\mathbb{I}\|_c,$$

which implies that $\mathcal{L}$ is continuous and uniformly bounded since $\sum_{j=0}^{\infty} e^{-\kappa\omega j}$ is convergent.

In the following we prove the compactness of $\mathcal{L}$. We first claim that $\mathcal{L}\mathbb{I}(t)$ is equicontinuous. Consider $\mathbb{I}(t)\in\mathbb{C}_\omega$ and $\forall t_1,t_2\in[0,\omega]$ with $t_1<t_2$. Then

$$\|\mathcal{L}\mathbb{I}(t_2)-\mathcal{L}\mathbb{I}(t_1)\|_1 = \left\|\int_{-\infty}^{t_2} Y(t_2,s)\mathbb{F}(s)\mathbb{I}(s)ds - \int_{-\infty}^{t_1} Y(t_1,s)\mathbb{F}(s)\mathbb{I}(s)ds\right\|_1$$
$$= \left\|\int_{-\infty}^{t_2} (Y(t_2,s)-Y(t_1,s))\mathbb{F}(s)\mathbb{I}(s)ds + \int_{t_1}^{t_2} Y(t_1,s)\mathbb{F}(s)\mathbb{I}(s)ds\right\|_1$$
$$\leq \int_{-\infty}^{t_2} \|Y(t_2,s)-Y(t_1,s)\|_1\|\mathbb{F}(s)\|_1\|\mathbb{I}(s)\|_1 ds + \int_{t_1}^{t_2} \|Y(t_1,s)\|_1\|\mathbb{F}(s)\|_1\|\mathbb{I}(s)\|_1 ds$$
$$\leq \int_{-\infty}^{\omega} \|Y(t_2,s)-Y(t_1,s)\|_1\|\mathbb{F}(s)\|_1\|\mathbb{I}(s)\|_1 ds + \int_{t_1}^{t_2} Ke^{-\kappa(t_1-s)}\|\mathbb{F}(s)\|_1\|\mathbb{I}(s)\|_1 ds$$
$$\leq \|e^{-Vt_2}-e^{-Vt_1}\|_1 \sum_{i=-\infty}^{0}\int_{i\omega}^{(i+1)\omega} K_1\|e^{Vs}\|_1\|\mathbb{I}(s)\|_1 ds + \int_{t_1}^{t_2} Ke^{-\kappa(t_1-s)}K_1\|\mathbb{I}(s)\|_1 ds$$
$$\leq \sum_{i=-\infty}^{0} e^{\tilde{d}_1(i+1)\omega}\cdot K_1\|\mathbb{I}\|_c\|e^{-Vt_2}-e^{-Vt_1}\|_1 + KK_1e^{\kappa\omega}\|\mathbb{I}\|_c(t_2-t_1),$$

where $\tilde{d}_1=\max\{d+r_a,d+r_s\}$.

Notice that $\sum_{i=-\infty}^{0} e^{N(i+1)}$ is convergent and e^{-Vt} is continuous on $[0,\omega]$. Thus, if $\{\mathbb{I}(t)\}$ is bounded, for $\forall\epsilon>0$ there exists a $\delta>0$ such that $\|\mathcal{L}\mathbb{I}(t_2)-\mathcal{L}\mathbb{I}(t_1)\|_c<\epsilon$ as $|t_2-t_1|<\delta$. This implies that $\{(\mathcal{L}\mathbb{I})(t)\}$ are equicontinuous. According to Ascoli-Arzela theorem, we know that $\mathcal{L}$ is compact. The proof of this lemma is completed. □

$\mathcal{L}$ is called the next infection operator, and the spectral radius of $\mathcal{L}$ can be defined as the basic reproduction number (or ratio)

$$\mathcal{R}_0 := \rho(\mathcal{L}) \tag{3.6}$$

of system (2.3).

Following [25], we consider how to calculate $\mathcal{R}_0$ and whether the basic reproduction ratio (or number) $\mathcal{R}_0$ characterizes the threshold of disease invasion, i.e., the disease-free periodic solution $(N,0,0)$ of system (2.3) is locally asymptotically stable if $\mathcal{R}_0<1$ and unstable if $\mathcal{R}_0>1$.

It is clear that the disease-free periodic solution $(N,0,0)$ of system (2.3) is locally asymptotically stable if all characteristic multipliers of periodic system (3.1) are

less than one, and it is unstable if at least one of the characteristic multipliers of periodic system (3.1) is greater than one. By straightforward calculation, we obtain that the characteristic multipliers of periodic system (3.1) consist of $e^{-(d+\sigma)\omega}$ and the eigenvalues of the following matrix

$$\Phi_{F-V}(\omega) = e^{(F_2-V)\theta\omega}e^{(F_1-V)(1-\theta)\omega},$$

where

$$F_i - V = \begin{pmatrix} \mu\beta_i N - (d+r_a) & \alpha\mu\beta_i N \\ (1-\mu)\beta_i N & \alpha(1-\mu)\beta_i N - (d+r_s) \end{pmatrix}, \quad i = 1, 2.$$

Note that $e^{-(d+\sigma)\omega} < 1$ because $d + \sigma > 0$. Therefore, all characteristic multipliers of periodic system (3.1) are less than one if and only if the largest eigenvalue of $\Phi_{F-V}(\omega)$, denoted by $\rho(\Phi_{F-V}(\omega))$, is less than one (i.e. $\rho(\Phi_{F-V}(\omega)) < 1$), and at least one of the characteristic multipliers of periodic system (3.1) is greater than one if and only if $\rho(\Phi_{F-V}(\omega)) > 1$, here $\rho(\Phi_{F-V}(\omega))$ is called *the spectral radius* of matrix $\Phi_{F-V}(\omega)$.

On the other hand, it is easy to check that all assumptions (A2)-(A7) in [25] are valid for system (3.1) except the assumption (A1). Using the notations in [25], we define a matrix $V_\varepsilon = V - \varepsilon P$, here $P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and ε is a very small positive number. Thus, $-V_\varepsilon$ is cooperative and irreducible for each $t \in \mathbb{R}$. Let $Y_\varepsilon(t, s)$ be the evolution operator of the linear system (3.3) with V replaced by V_ε. For some small ε_0, as $\varepsilon \in [0,\ \varepsilon_0)$ we can define the linear operator $\mathcal{L}_\varepsilon$ by replacing $Y(t, s)$ in (3.5) with $Y_\varepsilon(t, s)$ such that the operator $\mathcal{L}_\varepsilon$ is positive, continuous and compact on $\mathbb{C}_\omega$. Let $\mathcal{R}_0^\varepsilon := \rho(\mathcal{L}_\varepsilon)$ for $\varepsilon \in [0,\ \varepsilon_0)$.

By the proof of Theorem 2.1, we know that the solutions of the following system

$$\frac{dx}{dt} = (\mathbb{F}(t) - V_\varepsilon)x \tag{3.7}$$

are continuous with respect to all parameters. Thus,

$$\lim_{\varepsilon\to 0} \Phi_{F-V_\varepsilon}(\omega) = \Phi_{F-V}(\omega),$$

where $\Phi_{F-V_\varepsilon}(\omega)$ is the monodromy matrix of system (3.7), and $\Phi_{F-V}(\omega)$ is the monodromy matrix of system (3.7) as $\varepsilon = 0$.

According to the continuity of the spectrum of matrices, we have

$$\lim_{\varepsilon\to 0} \rho(\Phi_{F-V_\varepsilon}(\omega)) = \rho(\Phi_{F-V}(\omega)).$$

From Lemma 3.1, we use the similar arguments in [25] to the two linear operator $\mathcal{L}_\varepsilon$ and $\mathcal{L}$, and obtain that

$$\lim_{\varepsilon\to 0} \mathcal{R}_0^\varepsilon = \mathcal{R}_0.$$

We now easily follow the arguments in [25] to characterize $\mathcal{R}_0$. Let $W_\lambda(t, s), t \geq s$ be the fundamental solution matrix of the following linear periodic system

$$\frac{dw}{dt} = \left(-V + \frac{\mathbb{F}(t)}{\lambda}\right) w,$$

where the parameter $\lambda \in (0, +\infty)$. Consider an equation of λ

$$\rho(W_\lambda(\omega, 0)) = 1. \tag{3.8}$$

Then $\mathcal{R}_0$ can be calculated as follows.

Theorem 3.2. *(i) If equation (3.8) has a solution $\lambda_0 > 0$, then λ_0 is an eigenvalue of $\mathcal{L}$, which implies that $\mathcal{R}_0 > 0$;*
(ii) If $\mathcal{R}_0 > 0$, then $\lambda = \mathcal{R}_0$ is the only solution of equation (3.8);
(iii) $\mathcal{R}_0 = 0$ if and only if $\rho(W_\lambda(\omega, 0)) < 1$ for all positive λ.

Note that $\rho(W_1(\omega, 0)) = \rho(\Phi_{F-V}(\omega))$. Using similar arguments in [25], we can prove that the basic reproduction ratio (or number) $\mathcal{R}_0$ can characterize the threshold of disease invasion.

Theorem 3.3. *(i) $\mathcal{R}_0 > 1$ if and only if $\rho(\Phi_{F-V}(\omega)) > 1$;*
(ii) $\mathcal{R}_0 = 1$ if and only if $\rho(\Phi_{F-V}(\omega)) = 1$;
(iii) $\mathcal{R}_0 < 1$ if and only if $\rho(\Phi_{F-V}(\omega)) < 1$.
Hence, the disease-free periodic solution $(N, 0, 0)$ of system (2.3) is locally asymptotically stable if $\mathcal{R}_0 < 1$, and it is unstable if $\mathcal{R}_0 > 1$.

To save space, the proofs of the above theorems are omitted. From Theorem 3.3, we can see that $\mathcal{R}_0$ is a threshold parameter for local stability of the disease-free periodic solution $(N, 0, 0)$. We next show that $\mathcal{R}_0$ is also a threshold parameter for dynamics of system (2.3) in $\mathcal{D}_0$.

Theorem 3.4. *When $\mathcal{R}_0 < 1$, solutions $(S(t), I_a(t), I_s(t))$ of system (2.3) with initial points in $\mathcal{D}_0$ satisfies*

$$\lim_{t\to+\infty}(S(t), I_a(t), I_s(t)) = (N, 0, 0).$$

And the disease-free periodic solution $(N, 0, 0)$ of system (2.3) is globally asymptotically stable in $\mathcal{D}_0$.

Proof. In the invariant pyramid $\mathcal{D}_0$ as shown in (2.4), we consider a subsystem by the last two equations of system (2.3)

$$\begin{cases} \dot{I}_a(t) &= \mu\beta(t)S(I_a + \alpha I_s) - (d + r_a)I_a \\ &\le \mu\beta(t)N(I_a + \alpha I_s) - (d + r_a)I_a, \\ \dot{I}_s(t) &= (1-\mu)\beta(t)S(I_a + \alpha I_s) - (d + r_s)I_s \\ &\le (1-\mu)\beta(t)N(I_a + \alpha I_s) - (d + r_s)I_s. \end{cases} \tag{3.9}$$

Thus, the auxiliary system of (3.9) is

$$\begin{cases} \dot{I}_a(t) = \mu\beta(t)N(I_a + \alpha I_s) - (d + r_a)I_a, \\ \dot{I}_s(t) = (1-\mu)\beta(t)N(I_a + \alpha I_s) - (d + r_s)I_s, \end{cases} \tag{3.10}$$

which is a periodic linear discontinuous system with period ω. The periodic map associated with system (3.10) is defined by $\Phi_{F-V}(\omega)$, which is a linear continuous map.

When $\mathcal{R}_0 < 1$, we have $\rho(\Phi_{F-V}(\omega)) < 1$, which implies that $(0, 0)$ is a globally asymptotically stable solution of system (3.10).

Note that systems (3.9) and (3.10) are cooperative. Using the similar arguments in [18], we can prove that the comparison principle holds. Hence,

$$\lim_{t\to+\infty}(I_a(t), I_s(t)) = (0, 0).$$

So, for arbitrarily small constant $\varepsilon > 0$, there exists $T > 0$ such that $I_a(t) + \alpha I_s(t) < \varepsilon$ as $t > T$. From the first equation of system (2.3), we have

$$\begin{aligned} \dot{S} &= dN - dS - \beta(t)S(I_a + \alpha I_s) + \sigma(N - S - I_a - I_s) \\ &> dN - dS - \beta_2 S\varepsilon. \end{aligned}$$

Therefore, $\liminf_{t\to+\infty} S(t) \geq \frac{dN}{d+\beta_2\varepsilon}$. Let $\varepsilon \to 0$, we have

$$\liminf_{t\to+\infty} S(t) \geq N.$$

On the other hand, $S(t) \leq N$ in $\mathcal{D}_0$, which admits

$$\lim_{t\to+\infty} S(t) = N.$$

In summary, we have $\lim_{t\to+\infty}(S(t), I_a(t), I_s(t)) = (N, 0, 0)$. Moreover, from Theorem 3.3 we know that $(N, 0, 0)$ of system (2.3) is globally asymptotically stable. □

In the following, we show that the disease is uniformly persistent when $\mathcal{R}_0 > 1$.

Theorem 3.5. *If $\mathcal{R}_0 > 1$, $0 < \mu < 1$ and $0 < \alpha\beta_1$, then there exists a constant $\delta_0 > 0$ such that every solution $(S(t), I_a(t), I_s(t))$ of system (2.3) with initial values in $\mathcal{D}_0$ satisfies*

$$\liminf_{t\to+\infty} I_a(t) \geq \delta_0, \quad \liminf_{t\to+\infty} I_s(t) \geq \delta_0.$$

Proof. Since system (2.3) is ω-periodic with respect to t in $\mathbb{R}_+ \times \mathcal{D}_0$, it suffices to investigate the dynamics of its associated period map $\mathcal{P}$ defined by (2.8) on $\mathcal{D}_0$ for the dynamics of system (2.3), where the map $\mathcal{P}$ is continuous. Clearly, $\mathcal{P}(\mathcal{D}_0) \subset \mathcal{D}_0$. Define

$$X_0 = \{(S, I_a, I_s) \in \mathcal{D}_0 : I_a > 0, I_s > 0\}, \ \partial X_0 = \mathcal{D}_0 \backslash X_0.$$

Set

$$M_\partial = \{P_0 \in \partial X_0 : \mathcal{P}^k(P_0) \in \partial X_0, \forall k \geq 0\},$$

which is a positive invariant set of $\mathcal{P}$ in ∂X_0. We claim

$$M_\partial = \{(S, 0, 0) : 0 \leq S \leq N\}. \tag{3.11}$$

In fact, $\{(S, 0, 0) : 0 \leq S \leq N\} \subset M_\partial$ by (2.8). On the other hand, for any $P_0 \in \partial X_0 \setminus \{(S, 0, 0) : 0 \leq S \leq N\}$, that is, either $I_{a0} = 0, I_{s0} > 0, S_0 \geq 0$ or $I_{a0} > 0, I_{s0} = 0, S_0 \geq 0$. In the case $I_{a0} = 0, I_{s0} > 0, S_0 > 0$ (resp. $I_{a0} > 0, I_{s0} = 0, S_0 > 0$), we calculate by the last two equations of system (2.3) and obtain that

$$I_a'(0) = \mu\alpha\beta(0)S(0)I_s(0) > 0 \text{ (resp. } I_s'(0) = (1-\mu)\beta(0)S(0)I_a(0) > 0),$$

if $0 < \mu < 1$ and $0 < \alpha\beta_1$. This implies that $\mathcal{P}^{k_0}(P_0) \notin \partial X_0 \backslash \{(S, 0, 0) : 0 \leq S \leq N\}$ for some $k_0 \geq 0$ since the subsystem by the last two equations of system (2.3) is cooperative. If $S(0) = 0, I_{a0} = 0, I_{s0} > 0$ (or $S(0) = 0, I_{a0} > 0, I_{s0} = 0$), then $S'(0) = (d+\sigma)N - \sigma I_s(0) > 0$ (or $S'(0) = (d+\sigma)N - \sigma I_a(0) > 0$), which leads that $\mathcal{P}^{k_1}(P_0) \notin \partial X_0 \setminus \{(S, 0, 0) : 0 \leq S \leq N\}$ for some $k_1 \geq 0$. Therefore, (3.11) is proved and M_∂ is the maximal compact invariant set of $\mathcal{P}$ in ∂X_0.

Note that $E_0(N, 0, 0)$ is the unique fixed point of $\mathcal{P}$ in M_∂ and it is an attractor of $\mathcal{P}$ in M_∂ by the first equation of (2.3). Since $\mathcal{R}_0 > 1$, the stable set $W^s(E_0)$ of E_0 satisfies that $W^s(E_0) \cap X_0 = \emptyset$.

Applying [29, Theorem 1.3.1], we obtain that $\mathcal{P}$ is uniformly persistent with respect to $(X_0, \partial X_0)$. Moreover, from [29, Theorem 3.1.1], it can see that the conclusion of this theorem is true. The proof is completed. □

4. **Global dynamics of system (2.3) without seasonal force.** In this section, we study the effects of asymptomatic infection on the dynamics of system (2.3) if there are not seasonal factors, that is, $\beta_1 = \beta_2 = \beta$. Then system (2.3) becomes

$$\begin{cases} \dot{S} = (d+\sigma)(N-S) - \beta S(I_a + \alpha I_s) - \sigma(I_a + I_s), \\ \dot{I}_a = \mu\beta S(I_a + \alpha I_s) - (d + r_a)I_a, \\ \dot{I}_s = (1-\mu)\beta S(I_a + \alpha I_s) - (d + r_s)I_s \end{cases} \tag{4.1}$$

in the domain $\mathbb{R}^3_+$.

By the formula (3.6), we let $\beta_1 = \beta_2$ and obtain the basic reproduction number $\mathcal{R}_0$ of system (4.1) as follows.

$$\mathcal{R}_0 = \beta N \left(\frac{\mu}{d + r_a} + \frac{\alpha(1-\mu)}{d + r_s} \right), \tag{4.2}$$

which is consistent with the number calculated using the approach of basic reproduction number in [7] and [23].

From the expression (4.2), we can see that there is still the risks of infectious disease outbreaks due to the existence of asymptomatic infection even if all symptomatically infective individuals have been quarantined, that is, $\alpha = 0$. This provides an intuitive basis for understanding that the asymptomatically infective individuals promote the evolution of epidemic.

In the following we study the dynamics of system (4.1). By a straightforward calculation, we obtain the existence of equilibria for system (4.1).

Lemma 4.1. *System (4.1) has the following equilibria in $\mathbb{R}^3_+$.*

(i) If $\mathcal{R}_0 \le 1$, then system (4.1) has a unique equilibrium, which is the disease-free equilibrium $E_0(N, 0, 0)$.

(ii) If $\mathcal{R}_0 > 1$ and $0 < \mu < 1$, then system (4.1) has two equilibria: the disease-free equilibrium $E_0(N, 0, 0)$ and the endemic equilibrium $E_1(S^*, I_a^*, I_s^*)$ in the interior of $\mathcal{D}_0$, where $S^* = \frac{N}{\mathcal{R}_0}, I_a^* = \frac{\mu(d+\sigma)(d+r_s)N}{(d+r_a)(d+r_s)+\sigma(d+\mu r_s)+\sigma r_a(1-\mu)}(1 - \frac{1}{\mathcal{R}_0}), I_s^* = \frac{(1-\mu)(d+r_a)}{\mu(d+r_s)} I_a^*$.

(iii) If $\mathcal{R}_0 > 1$ and $\mu = 0$, then system (4.1) has two equilibria: the disease-free equilibrium $E_0(N, 0, 0)$ and the asymptomatic-free equilibrium $E_2(S_2^*, 0, I_{s2}^*)$, where $S_2^* = \frac{N}{\mathcal{R}_0}, I_{s2}^* = \frac{d+\sigma}{d+\sigma+r_s} N(1 - \frac{1}{\mathcal{R}_0})$.

(iv) If $\mathcal{R}_0 > 1$ and $\mu = 1$, the system (4.1) has two equilibria: the disease-free equilibrium $E_0(N, 0, 0)$ and the symptomatic-free equilibrium $E_3(S_3^*, I_{a3}^*, 0)$, where $S_3^* = \frac{N}{\mathcal{R}_0}, I_{a3}^* = \frac{d+\sigma}{d+\sigma+r_a} N(1 - \frac{1}{\mathcal{R}_0})$.

We now discuss the local stability and topological classification of these equilibria in $\mathbb{R}^3_+$, respectively. We first study the disease-free equilibrium $E_0(N, 0, 0)$ and have the following lemma.

Lemma 4.2. *The disease-free equilibrium $E_0(N, 0, 0)$ of system (4.1) in $\mathbb{R}^3_+$ is asymptotically stable if $\mathcal{R}_0 < 1$; $E_0(N, 0, 0)$ is a saddle-node with one dimensional center manifold and two dimensional stable manifold if $\mathcal{R}_0 = 1$; and $E_0(N, 0, 0)$ is a saddle with two dimensional stable manifold and one dimensional unstable manifold if $\mathcal{R}_0 > 1$.*

Proof. A routine computation shows that the characteristic polynomial of system (4.1) at E_0 is

$$f_1(\lambda) = (\lambda + d + \sigma)(\lambda^2 - a_1\lambda + a_0), \tag{4.3}$$

where $a_0 = (d + r_a)(d + r_s)(1 - \mathcal{R}_0)$,

$$a_1 = (d + r_a)(\beta N \frac{\mu}{d + r_a} - 1) + (d + r_s)(\alpha\beta N \frac{1 - \mu}{d + r_s} - 1).$$

It is clear that $-(d + \sigma) < 0$ is always one root of (4.3). We consider three cases, $\mathcal{R}_0 < 1$, $\mathcal{R}_0 = 1$ and $\mathcal{R}_0 > 1$, to discuss the other roots of (4.3).

If $\mathcal{R}_0 < 1$, then $a_1 < 0$ and $a_0 > 0$ by $\beta N \frac{\mu}{d+r_a} < \mathcal{R}_0$ and $\beta N \frac{\alpha(1-\mu)}{d+r_s} < \mathcal{R}_0$. Thus, three roots of (4.3) have negative real parts, which leads to the local asymptotical stability of the disease-free equilibrium E_0.

If $\mathcal{R}_0 = 1$, then $a_0 = 0$ and $a_1 < 0$. Hence, the characteristic equation $f_1(\lambda) = 0$ has three roots: $\lambda_1 = -(d + \sigma) < 0$, $\lambda_2 = a_1 < 0$ and $\lambda_3 = 0$. For calculating the associated eigenvectors v_i of λ_i, $i = 1, 2, 3$, we consider $J(E_0)$ with respect to μ in three cases: (i) $0 < \mu < 1$, (ii) $\mu = 0$ and (iii) $\mu = 1$, and we can obtain that E_0 is a saddle-node with one dimensional center manifold and two dimensional stable manifold by tedious calculations of normal form.

Summarized the above analysis, we complete the proof of this lemma. □

From Lemma 4.1 and Lemma 4.2, we can see that system (4.1) undergoes saddle-node bifurcation in a small neighborhood of $E_0(N, 0, 0)$ as $\mathcal{R}_0$ increases passing through $\mathcal{R}_0 = 1$.

About the endemic equilibria, we have the following local stability.

Lemma 4.3. *The endemic equilibrium $E_1(S^*, I_a^*, I_s^*)$ of system (4.1) is asymptotically stable if $\mathcal{R}_0 > 1$ and $0 < \mu < 1$; the asymptomatic-free equilibrium $E_2(S_2^*, 0, I_{s2}^*)$ of system (4.1) is asymptotically stable if $\mathcal{R}_0 > 1$ and $\mu = 0$; and the symptomatic-free equilibrium $E_3(S_3^*, I_{a3}^*, 0)$ of system (4.1) is asymptotically stable if $\mathcal{R}_0 > 1$ and $\mu = 1$.*

Proof. Either $\mu = 0$ or $\mu = 1$, it is easy to compute the eigenvalues of the Jacobian matrix of system (4.1) at E_2 or E_3, respectively, and find that all eigenvalues have negative real parts. Hence, $E_2(S_2^*, 0, I_{s2}^*)$ or $E_3(S_3^*, I_{a3}^*, 0)$ is asymptotically stable if $\mathcal{R}_0 > 1$, respectively.

After here we only prove that $E_1(S^*, I_a^*, I_s^*)$ is asymptotically stable if $\mathcal{R}_0 > 1$ and $0 < \mu < 1$. To make the calculation easier, we use the change of variables

$$S = \frac{(d + r_s)}{\mu\beta}\hat{S},\ I_a = \frac{(d + r_s)}{\beta}\hat{I}_a,\ I_s = \frac{(d + r_s)}{\beta}\hat{I}_s,\ dt = \frac{d\tau}{(d + r_s)},$$

which reduces system (4.1) into the following system,

$$\begin{cases} \frac{dS}{d\tau} = N_1 - d_1 S - \sigma_1 I_a - \sigma_1 I_s - S(I_a + \alpha I_s), \\ \frac{dI_a}{d\tau} = -r I_a + S(I_a + \alpha I_s), \\ \frac{dI_s}{d\tau} = -I_s + \mu_1 S(I_a + \alpha I_s), \end{cases} \tag{4.4}$$

where

$$N_1 = N(d + \sigma)\mu\beta/(d + r_s)^2,\ d_1 = (d + \sigma)/(d + r_s),$$
$$\sigma_1 = \sigma\mu/(d + r_s),\ r = (d + r_a)/(d + r_s),\ \mu_1 = (1 - \mu)/\mu$$

and for simplicity we denote $\hat{S}, \hat{I}_a, \hat{I}_s$ by S, I_a, I_s respectively.

When $\mathcal{R}_0 > 1$, the disease-free equilibrium $E_0(N, 0, 0)$ and endemic equilibrium $E_1(S^*, I_a^*, I_s^*)$ of system (4.1) are transformed into the disease-free equilibrium

$\hat{E}_0(N_1/d_1, 0, 0)$ and endemic equilibrium $\hat{E}_1(\hat{S}^*, \hat{I}_a^*, \hat{I}_s^*)$ of system (4.4) respectively, where

$$\hat{S}^* = \frac{N_1/d_1}{\hat{R}_0},\ \hat{I}_a^* = \frac{N_1}{\sigma_1 + r\sigma_1\mu_1 + r}(1 - \frac{1}{\hat{R}_0}),\ \hat{I}_s^* = \mu_1 r I_a^*.$$

Notice that $\hat{R}_0 := \frac{N_1}{d_1}(\frac{1}{r} + \alpha\mu_1) > 1$ if and only if $\mathcal{R}_0 > 1$.

The characteristic equation of system (4.4) at $\hat{E}_1$ is

$$f_2(\lambda) = \det(\lambda I - J(\hat{E}_1)) = \lambda^3 + \xi_2\lambda^2 + \xi_1\lambda + \xi_0,$$

where

$$\begin{aligned}
\xi_2 &= \{\sigma_1 + r\sigma_1\mu_1 + r + r^2\mu_1\alpha\sigma_1 + r^3\mu_1^2\alpha\sigma_1 + r^3\mu_1\alpha + d_1\sigma_1\mu_1 r\alpha + d_1\sigma_1\mu_1^2 r^2\alpha \\
&\quad + N_1 + d_1\sigma_1 + d_1 r\sigma_1\mu_1 + 2N_1\mu_1 r\alpha + r^2\mu_1^2\alpha^2 N_1\}/\{(\sigma_1 + r\sigma_1\mu_1 + r)(r\mu_1\alpha + 1)\}, \\
\xi_1 &= d_1(1 + r^2\mu_1\alpha)/(r\mu_1\alpha + 1) + (\sigma_1\mu_1 + 1 + r + \sigma_1)(r\mu_1\alpha + 1)\hat{I}_a^*, \\
\xi_0 &= N_1\mu_1 r\alpha + N_1 - rd_1 = rd_1(\hat{R}_0 - 1).
\end{aligned}$$

It can be seen that all coefficients ξ_j of polynomial $f_2(\lambda)$ are positive if $\hat{R}_0 > 1$, where $j = 0, 1, 2$. Moreover, we claim that $\xi_2\xi_1 - \xi_0 > 0$. In fact,

$$\xi_2\xi_1 - \xi_0 = c_0 + c_1\hat{I}_a^* + c_2(\hat{I}_a^*)^2,$$

where

$$\begin{aligned}
c_0 &= \frac{d_1(1 + r^2\mu_1\alpha)(r^2\mu_1\alpha + d_1\mu_1 r\alpha + 1 + d_1)}{(r\mu_1\alpha + 1)^2}, \\
c_1 &= d_1\mu_1^2 r\alpha\sigma_1 + r^3\mu_1\alpha + r^2\mu_1\alpha\sigma_1 + 2d_1 r^2\mu_1\alpha + d_1\sigma_1\mu_1 r\alpha + \sigma_1\mu_1 \\
&\quad + d_1\sigma_1\mu_1 + 1 + 2d_1 + d_1\sigma_1 + r(d_1 - \sigma_1\mu_1) + \mu_1 r\alpha(d_1 - \sigma_1), \\
c_2 &= (r\mu_1\alpha + 1)^2(\sigma_1\mu_1 + 1 + r + \sigma_1).
\end{aligned}$$

It is easy to see that $c_0 > 0$ and $c_2 > 0$. Note that $d_1 - \sigma_1\mu_1 = \frac{d + \sigma\mu}{d + r_s} > 0$ and $d_1 - \sigma_1 = \frac{d + \sigma(1-\mu)}{d + r_s} > 0$ since $0 < \mu < 1$. This implies that $c_1 > 0$. Moreover, $\hat{I}_a^* > 0$ yields that $\xi_2\xi_1 - \xi_0 > 0$ and what we claimed is proved.

By the Routh-Hurwitz Criterion, we know that all eigenvalues of the characteristic polynomial $f_2(\lambda)$ have negative real parts. Thus, the endemic equilibrium $\hat{E}_1$ of system (4.4) is asymptotically stable. This leads that the endemic equilibrium $E_1(S^*, I_a^*, I_s^*)$ of system (4.1) is also asymptotically stable. □

From Lemma 4.2 and Lemma 4.3, we can see that $\mathcal{R}_0$ is the threshold quantity of local dynamics of system (4.1) in $\mathbb{R}^3_+$. By Theorem 2.1, we only need to consider system (4.1) for its global dynamics in $\mathcal{D}_0$. The following theorems will show that $\mathcal{R}_0$ is also the threshold quantity of global dynamics of system (4.1) in $\mathcal{D}_0$.

Theorem 4.4. *If $\mathcal{R}_0 \leq 1$, then the disease-free equilibrium $E_0(N, 0, 0)$ of system (4.1) is globally asymptotically stable in $\mathcal{D}_0$.*

The proof of this theorem can be finished by constructing a Liapunov function

$$L(S, I_a, I_s) = I_a(t) + \frac{d + r_a}{d + r_s}\alpha I_s(t)$$

in $\mathcal{D}_0$. For the sake of simplicity, we omit the details.

Theorem 4.5. *If $\mathcal{R}_0 > 1$ and $\mu = 0$ (resp. $\mu = 1$), then $E_2(S_2^*, 0, I_{s2}^*)$ (resp. $E_3(S_3^*, I_{a3}^*, 0)$) attracts all orbits of system (4.1) in $\mathcal{D}_0$ except both $E_0(N, 0, 0)$ and a positive orbit γ in its two dimensional stable manifold, where*

$$\gamma = \{(S, I_a, I_s) \in \mathcal{D}_0 : \ I_a = 0, \ I_s = 0, \ 0 < S < N\}.$$

Proof. We first prove the case that $\mathcal{R}_0 > 1$ and $\mu = 0$. When $\mu = 0$, system (4.1) becomes

$$\begin{cases} \dot{S} = (d+\sigma)(N-S) - \beta S(I_a + \alpha I_s) - \sigma(I_a + I_s), \\ \dot{I}_a = -(d + r_a)I_a, \\ \dot{I}_s = \beta S(I_a + \alpha I_s) - (d + r_s)I_s. \end{cases} \tag{4.5}$$

It is clear that $\lim_{t\to+\infty} I_a(t) = 0$. Hence, the limit system of system (4.5) in $\mathcal{D}_0$ is

$$\begin{cases} \dot{S} = (d+\sigma)(N-S) - \alpha\beta S I_s - \sigma I_s, \\ \dot{I}_s = \alpha\beta S I_s - (d + r_s)I_s \end{cases} \tag{4.6}$$

in $\mathcal{D}_1 = \{(S, I_s) : \ 0 \le S \le N, \ 0 \le I_s \le N\}$, which has two equilibria: $(N, 0)$ and (S_2^*, I_{s2}^*). Equilibrium $(N, 0)$ is a saddle and (S_2^*, I_{s2}^*) is locally asymptotically stable if $\mathcal{R}_0 > 1$.

In the following we prove that (S_2^*, I_{s2}^*) attracts all orbits of system (4.6) in $\mathcal{D}_1$ except both $(N, 0)$ and its one dimensional stable manifold.

Let $x = S + \frac{\sigma}{\alpha\beta}$ and $y = I_s$. Then system (4.6) becomes

$$\begin{cases} \dot{x} = (d+\sigma)(N + \frac{\sigma}{\alpha\beta}) - (d+\sigma)x - \alpha\beta xy, \\ \dot{y} = \alpha\beta xy - (d + r_s + \sigma)y. \end{cases} \tag{4.7}$$

Hence, $(x_0, y_0) = (S_2^* + \frac{\sigma}{\alpha\beta}, I_{s2}^*)$ is the unique positive equilibrium of system (4.7) if $\mathcal{R}_0 > 1$. Consider the Liapunov function of system (4.7)

$$V(x, y) = \frac{1}{2}(x - x_0)^2 + x_0\left(y - y_0 - y_0 \ln \frac{y}{y_0}\right)$$

in $\tilde{\mathcal{D}}_1 = \{(x, y) : \ \frac{\sigma}{\alpha\beta} \le x \le N + \frac{\sigma}{\alpha\beta}, \ 0 \le y \le N\}$. It is clear that $V(x, y) \ge 0$ and $V(x, y) = 0$ if and only if $x = x_0$ and $y = y_0$ in $\tilde{\mathcal{D}}_1$. And

$$\frac{dV(x(t), y(t))}{dt}|_{(4.7)} = -(x - x_0)^2(\alpha\beta y + d + \sigma) \le 0$$

in $\tilde{\mathcal{D}}_1$.

By LaSalle's Invariance Principle, we know that (x_0, y_0) attracts all orbits of system (4.7) in $\tilde{\mathcal{D}}_1$ except both equilibrium $(N + \frac{\sigma}{\alpha\beta}, 0)$ and its one dimensional stable manifold $\{(x, y) : \ y = 0, 0 < x < N + \frac{\sigma}{\alpha\beta}\}$. This leads to the conclusion, $E_2(S_2^*, 0, I_{s2}^*)$ attracts all orbits of system (4.1) in $\mathcal{D}_0$ except both $E_0(N, 0, 0)$ and a positive orbit γ if $\mathcal{R}_0 > 1$ and $\mu = 0$.

Using the similar arguments, we can prove that $E_3(S_3^*, I_{a3}^*, 0)$ attracts all orbits of system (4.1) in $\mathcal{D}_0$ except both $E_0(N, 0, 0)$ and a positive orbit γ if $\mathcal{R}_0 > 1$ and $\mu = 1$. □

Theorem 4.6. *If $\mathcal{R}_0 > 1$, $0 < \mu < 1$ and $r_a = r_s$, then the endemic equilibrium $E_1(S^*, I_a^*, I_s^*)$ attracts all orbits of system (4.1) in $\mathcal{D}_0$ except $E_0(N, 0, 0)$.*

Proof. Let $I = I_a + \alpha I_s,\ N_1 = S + I_a + I_s$. Then under the assumption $r_a = r_s = r$, system (4.1) in $\mathcal{D}_0$ can be written as

$$\begin{cases} \dot{S} = (d+\sigma)N - \sigma N_1 - dS - \beta SI, \\ \dot{I} = \tilde{\mu} SI - (d+r)I, \\ \dot{N}_1 = (d+\sigma)N - (d+r+\sigma)N_1 + rS \end{cases} \tag{4.8}$$

in $\tilde{\mathcal{D}}_0 := \{(S, I, N_1) |\ S \geq 0, I \geq 0,\ N \geq N_1 \geq 0\}$, where $\tilde{\mu} = (\mu + \alpha(1-\mu))\beta$.

Thus, equilibrium $E_1(S^*, I_a^*, I_s^*)$ of system (4.1) becomes equilibrium $\tilde{E}_1(S^*, I^*, N_1^*)$ of system (4.8) and $\tilde{E}_1$ is locally asymptotically stable, where $I^* = I_a^* + \alpha I_s^*, N_1^* = S^* + I_a^* + I_s^*$.

Applying a typical approach of Liapunov functions, we define

$$g(x) = x - 1 - \ln x,$$

and construct a Liapunov function of system (4.8)

$$V_1(S, I, N_1) = \frac{\nu_1}{2}(S - S^*)^2 + \nu_2 I^* g(\frac{I}{I^*}) + \frac{\nu_3}{2}(N_1 - N_1^*)^2,$$

where arbitrary constants $\nu_1 > 0$, $\nu_2 = \nu_1 \beta S^*/\tilde{\mu}$ and $\nu_3 = \nu_1 \sigma / r$. Note that $g(x) \geq g(1) = 0$ for all $x > 0$ and the global minimum $g(x) = 0$ is attained if and only if $x = 1$. Thus, $V_1(S, I, N_1) \geq 0$ and $V_1(S, I, N_1) = 0$ if and only if $S = S^*$, $I = I^*$ and $N_1 = N_1^*$ in $\tilde{\mathcal{D}}_0$.

The derivative of V_1 along the trajectories of system (4.8) is

$$\begin{aligned} \frac{dV_1(S, I, N_1)}{dt} =& -\nu_1 d {S^*}^2 (x-1)^2 - \nu_3 (d+r+\sigma) {N_1^*}^2 (z-1)^2 \\ & - \nu_1 \beta {S^*}^2 I^* y (x-1)^2 \leq 0, \end{aligned}$$

where $x = \frac{S}{S^*},\ y = \frac{I}{I^*},\ z = \frac{N_1}{N_1^*}$.

Note that the only compact invariant subset of the set $\{(S, I, N_1) : \frac{dV_1(S,I,N_1)}{dt} = 0\}$ is the singleton $\tilde{E}_1(S^*, I^*, N_1^*)$ in $\tilde{\mathcal{D}}_0$. Consequently, we can conclude that $E_1(S^*, I_a^*, I_s^*)$ is globally asymptotically stable and attracts all orbits of system (4.1) in $\mathcal{D}_0$ except $E_0(N, 0, 0)$. □

From Theorem (4.6) and the continuity of solutions with respect to parameters r_a and r_s, we obtain the following results.

Theorem 4.7. *If $\mathcal{R}_0 > 1$ and $0 < \mu < 1$, then the endemic equilibrium $E_1(S^*, I_a^*, I_s^*)$ is globally asymptotical stable in the interior of $\mathcal{D}_0$ for $0 < |r_s - r_a| \ll 1$.*

5. **Discussion.** In this paper, we established a compartmental SIRS epidemic model with asymptomatic infection and seasonal factors. In our model, we divided the period of the disease transmission into two seasons. In fact, it can be divided into n seasons for any given $n \in \mathbb{Z}_+$. Compared with continuous periodic systems, our piecewise continuous periodic model can provide a straightforward method to evaluate the basic reproduction number $\mathcal{R}_0$, that is to calculate the spectral radius of the matrix $\Phi_{F-V}(\omega) = e^{(F_2 - V)\theta\omega} e^{(F_1 - V)(1-\theta)\omega}$. It is shown that the length of the season, the transmission rate and the existence of asymptomatic infective affect the basic reproduction number $\mathcal{R}_0$, and there is still the risks of infectious disease outbreaks due to the existence of asymptomatic infection even if all symptomatically infective individuals have been quarantined, that is, $\alpha = 0$. This provides an intuitive basis for understanding that the asymptomatically infective individuals

and the disease seasonal transmission promote the evolution of the epidemic. And theoretical dynamics of the model allow us to predict outcomes of control strategies during the course of the epidemic.

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