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Isochronicity and linearizability of a planar cubic system

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A B S T R A C T

In this paper we investigate the problem of linearizability for a family of cubic complex planar systems of ordinary differential equations. We give a classification of linearizable systems in the family obtaining conditions for linearizability in terms of parameters. We also discuss coexistence of isochronous centers in the systems.

1. Introduction

For planar real analytic differential systems of the form

$$\dot{x} = -y + P(x, y), \qquad \dot{y} = x + Q(x, y), \tag{1.1}$$

where P and Q are polynomials without constant and linear terms, it is well known that the origin can be either a center or a focus. In the first case all solutions in a neighbourhood of the origin are periodic and their trajectories are closed curves. If the origin is a center, there arises the problem to determine whether all periodic solutions in a neighbourhood of the origin have the same period. This problem is known as the isochronicity problem.

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The studies of isochronicity of polynomial differential systems go back to Lloud [23], who found the necessary and sufficient conditions for isochronicity of system (1.1) when P and Q are homogeneous polynomials of degree two. Latter on, Pleshkan [25] solved the isochronicity problem in the case when P and Q are homogeneous polynomials of degree three (see also [20]). In the case when P and Q are homogeneous polynomials of degree five the problem was solved in [26], however, the case of homogeneous polynomial of degree four is still unsolved, and some partial results can be found in [6,10]. A number of works is devoted to the investigation of some other particular families (see, e.g. [3,6–9,11,18,21,22,24,28,30] and references given there).

The following family of planar cubic systems

$$\begin{aligned} \dot{x} &= -y + p_2(x,y) + xr_2(x,y) = P(x,y), \\ \dot{y} &= x + q_2(x,y) + yr_2(x,y) = Q(x,y), \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} p_2 &= a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ q_2 &= b_{20}x^2 + b_{11}xy + b_{02}y^2, \\ r_2 &= r_{20}x^2 + r_{11}xy + r_{02}y^2, \end{aligned}$$

has been studied in [4,5,19,22] for the case when all parameters are real.

In [4] and [5] the authors have shown that real system (1.2) has a center and an isochronous center, respectively, if and only if in polar coordinates after some transformations it can be written in one of four and five forms, respectively. However from their results it is difficult to determine the conditions on parameters of polynomials p_2, q_2, r_2 for existence of centers and isochronous centers. Conditions on parameters of p_2, q_2, r_2 for the existence of a center were obtained in [22] and later on using another approach in [19].

In the work [22] published in 1997 the authors obtained the necessary and sufficient conditions for existence of isochronous center of system (1.2) represented by four series of condition on coefficients of the system, however in the more recent paper [5] published in 1999 the authors gave five conditions for existence of isochronous center of system (1.2). One of aims of this paper is to clarify the conditions for isochronicity of system (1.2). For this purpose we use an approach different from the ones of [22] and [5], namely we consider system (1.2) as system with complex coefficients and find conditions for linearization of the system. We obtain five series of conditions for linearizability of (1.2) and show that all linearizable systems are Darboux linearizable.

The paper is organized as follows. In Section 2 we recall some definitions and describe briefly a procedure to study the isochronicity and linearizability of polynomial systems. Applying this procedure, in Section 3 we present our main result, Theorem 3.1, which gives conditions for linearizability of system (1.2). In Section 4 we present the relation between the results obtained in Theorem 3.1 (and in [22]) and the results of [5]. Finally, in the last section we discuss the coexistence of isochronous centers in system (1.2).

2. Linearizability quantities and Darboux linearization

In this section we remind some statements related to isochronicity and linearizability of polynomial differential systems and describe an approach to compute the linearizability quantities for the system

$$\dot{x} = -y + \sum_{p+q\geq 2}^{n} a_{p,q}x^p y^q = P(x,y), \quad \dot{y} = x + \sum_{p+q\geq 2}^{n} b_{p,q}x^p y^q = Q(x,y), \tag{2.1}$$

where $a_{p,q}$, $b_{p,q}$ are real or complex parameters.

If the equilibrium point at the origin of real system (2.1) is known to be a center it is said that this center is *isochronous* if all periodic solutions of (2.1) in a neighbourhood of the origin have the same period. System (2.1) is said to be *linearizable* if there is an analytic change of coordinates

$$x_1 = x + \sum_{m+n\geq 2} c_{m,n}x^m y^n := H_1(x,y), \qquad y_1 = y + \sum_{m+n\geq 2} d_{m,n}x^m y^n := H_2(x,y), \tag{2.2}$$

that reduces (2.1) to the linear system $\dot{x}_1 = -y_1$, $\dot{y}_1 = x_1$.

The following theorem, which goes back to Poincaré and Lyapunov, shows that the linearizability and isochronicity problems are equivalent. A proof can be found e.g. in [28].

Theorem 2.1. *The origin of real system (2.1) is an isochronous center if and only if the system is linearizable.*

One can compute isochronicity quantities (obstacles for isochronicity) either in polar or cartesian coordinates. Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ one first passes from system (2.1) to a system of the form

$$\dot{r} = r^2 R(r, \cos\theta, \sin\theta), \qquad \dot{\theta} = 1 - r\Theta(r, \cos\theta, \sin\theta)$$

and then either computes the period function directly (see e.g. [1]) or following the approach of [5] looks for a function

$$H(r,\theta) = H_1(\theta)r + H_2(\theta)r^2 + H_3(\theta)r^3 + \dots, \tag{2.3}$$

where $H_1 = \cos\theta$ and H_k are homogeneous trigonometric polynomials of degree k, such that

$$\ddot{H} + H \equiv 0, \tag{2.4}$$

then the obstacles for fulfilment of (2.4) are isochronicity quantities (also called isochronous constants).

Using the cartesian coordinates it is convenient to write real system (2.1) in the complex form

$$\dot{z} = iz + Z(z,\bar{z}), \tag{2.5}$$

introducing the change $z = x + iy$ and then to look for a linearization of equation (2.5) (the approach used in [22]).

Since we would like to perform the investigation differently from [5] and [22] we use another computational approach. Namely, we look for conditions for linearizability of system (2.1) arising from applying transformation (2.2).

Taking the derivatives with respect to t on both sides of each equation of (2.2) we obtain

$$\dot{x}_1 = \dot{x} + \left(\sum_{m+n\geq 2} m c_{m,n}x^{m-1}y^n\right)\dot{x} + \left(\sum_{m+n\geq 2} n c_{m,n}x^m y^{n-1}\right)\dot{y},$$
$$\dot{y}_1 = \dot{y} + \left(\sum_{m+n\geq 2} m d_{m,n}x^{m-1}y^n\right)\dot{x} + \left(\sum_{m+n\geq 2} n d_{m,n}x^m y^{n-1}\right)\dot{y}.$$

Hence, the change of coordinates (2.2) linearizes system (2.1) if it holds that

$$\begin{aligned}
&\sum_{m+n\geq 2} d_{m,n}x^m y^n + \sum_{p+q\geq 2}^{n} a_{p,q}x^p y^q + \left(\sum_{m+n\geq 2} m c_{m,n}x^{m-1}y^n\right)\left(-y+\sum_{p+q\geq 2}^{n} a_{p,q}x^p y^q\right)\\
&+\left(\sum_{m+n\geq 2} n c_{m,n}x^m y^{n-1}\right)\left(x+\sum_{p+q\geq 2}^{n} b_{p,q}x^p y^q\right)\equiv 0,\\
&-\sum_{m+n\geq 2} c_{m,n}x^m y^n + \sum_{p+q\geq 2}^{n} b_{p,q}x^p y^q + \left(\sum_{m+n\geq 2} m d_{m,n}x^{m-1}y^n\right)\left(-y+\sum_{p+q\geq 2}^{n} a_{p,q}x^p y^q\right)\\
&+\left(\sum_{m+n\geq 2} n d_{m,n}x^m y^{n-1}\right)\left(x+\sum_{p+q\geq 2}^{n} b_{p,q}x^p y^q\right)\equiv 0.
\end{aligned}\tag{2.6}$$

Obstacles for the fulfilment of equations in (2.6) give us necessary conditions for existence of a linearizing change of coordinates (2.2) of system (2.1). Thus, a computational procedure to find necessary conditions for linearizability can be described as follows.

(1) Write the left hand sides of two equations in (2.6) in the form $\sum_{k,l\geq 2} h_1^{(k,l)}x^k y^l$, and $\sum_{k,l\geq 2} h_2^{(k,l)}x^k y^l$, respectively, where $h_1^{(k,l)}$ and $h_2^{(k,l)}$ are polynomials in the parameters $a_{p,q}, b_{p,q}$ $(p+q\geq 2)$ of system (2.1) and $c_{m,n}$, $d_{m,n}$ $(m+n\geq 2)$ of (2.2).

(2) Solve the polynomial system $h_i^{(k,l)}=0$ $(i=1,2,\ k+l=2)$ for the coefficients $c_{m,n}$, $d_{m,n}$ $(m+n=2)$ of (2.2).

(3) Solve the polynomial system $h_i^{(k,l)}=0$ $(i=1,2,\ k+l=3)$ for the coefficients $c_{m,n}$, $d_{m,n}$ $(m+n=3)$ of (2.2). In general case the system cannot be solved. However dropping from it two suitable equations we obtain a system that has a solution. We denote the two dropped polynomials on the left hand sides of these two equations by i_1 and j_1.

(4) Proceed step-by-step solving the polynomial systems $h_i^{(k,l)}=0$ $(i=1,2,\ k+l=r,\ r>3)$. Generally speaking, at all steps when $r=k+l$ is an odd number the polynomial system $h_i^{(k,l)}=0$ $(i=1,2,\ k+l=r)$ cannot be solved. Dropping on each such step two suitable equations (and denoting by $i_{(r-1)/2}$ and $j_{(r-1)/2}$ the corresponding polynomials), we obtain a system that has a solution.

This procedure yields the polynomials i_k and j_k which are polynomials in the parameters $a_{p,q}$ and $b_{p,q}$ of system (2.1) called the *linearizability quantities*. It is clear that system (2.1) admits a linearizing change of coordinates (2.2) if and only if $i_k=j_k=0$ for all $k>1$. Thus, the simultaneous vanishing of all linearizability quantities provides conditions which characterize when the system (2.1) is linearizable (equivalently it has an isochronous center at the origin). The ideal $\mathcal{L}=\langle i_1,j_1,i_2,j_2,...\rangle\subset\mathbb{C}[a,b]$ defined by the linearizability quantities is called the *linearizability ideal* and its affine variety, $V_{\mathcal{L}}=\mathbf{V}(\mathcal{L})$, is called the *linearizability variety*. Therefore, the linearizability problem will be solved finding the variety $V_{\mathcal{L}}$.

By the Hilbert Basis Theorem there exists a positive integer k such that $\mathcal{L}=\mathcal{L}_k=\langle i_1,j_1,...,i_k,j_k\rangle$. Note that the inclusion $V_{\mathcal{L}}\subset\mathbf{V}(\mathcal{L}_k)$ holds for any $k\geq 1$. The opposite inclusion is verified finding the irreducible decomposition of the variety $\mathbf{V}(\mathcal{L}_k)$ and then checking that any point of each component of the decomposition corresponds to a linearizable system. The irreducible decomposition can be found using the routine `minAssGTZ` [14] (which is based on the algorithm of [15]) of the computer algebra system Singular [13], however it involves extremely laborious calculations.

One of the most efficient methods to find a linearizing change of coordinates is the Darboux linearization method. To construct a Darboux linearization it is convenient to perform the substitution

$$z=x+iy,\qquad w=x-iy\tag{2.7}$$

obtaining from (2.1) a system of the form

$$\dot{z} = i(z + X(z,w)), \qquad \dot{w} = -i(w + Y(z,w)),$$

and, after the rescaling of time by i, the system

$$\dot{z} = z + X(z,w), \qquad \dot{w} = -w - Y(z,w). \tag{2.8}$$

Since the change of coordinates (2.7) is analytic, system (2.1) is linearizable if and only if system (2.8) is linearizable.

We remind that a *Darboux factor* of system (2.8) is a polynomial $f(z,w)$ satisfying

$$\frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial w}\dot{w} = Kf,$$

where $K(z,w)$ is a polynomial called the *cofactor of* f. A *Darboux linearization* of system (2.8) is an analytic change of coordinates $z_1 = Z_1(z,w)$, $w_1 = W_1(z,w)$, such that

$$Z_1(z,w) = \prod_{j=0}^{m} f_j^{\alpha_j}(z,w) = z + \tilde{Z}_1(z,w), \quad W_1(z,w) = \prod_{j=0}^{n} g_j^{\beta_j}(z,w) = w + \tilde{W}_1(z,w),$$

which linearizes (2.8), where $f_j, g_j \in \mathbb{C}[z,w]$, $\alpha_j, \beta_j \in \mathbb{C}$, and $\tilde{Z}_1$ and $\tilde{W}_1$ have neither constant nor linear terms. System (2.8) is said to be Darboux linearizable if it admits a Darboux linearization. The next theorem provides a way to construct a Darboux linearization using Darboux factors (see e.g. [28] for a proof).

Theorem 2.2. *System (2.8) is Darboux linearizable if and only if there exist $s+1 \geq 1$ Darboux factors $f_0, ..., f_s$ with corresponding cofactors $K_0, ..., K_s$ and $t+1 \geq 1$ Darboux factors $g_0, ..., g_t$ with corresponding cofactors $L_0, ..., L_t$ with the following properties:*

a. $f_0(z,w) = z + \cdots$ *but* $f_j(0,0) = 1$ *for* $j \geq 1$*;*
b. $g_0(z,w) = w + \cdots$ *but* $g_j(0,0) = 1$ *for* $j \geq 1$*; and*
c. *there are $s+t$ constants $\alpha_1, ..., \alpha_s, \beta_1, ..., \beta_t \in \mathbb{C}$ such that*

$$K_0 + \alpha_1 K_1 + \cdots + \alpha_s K_s = 1 \quad \text{and} \quad L_0 + \beta_1 L_1 + \cdots + \beta_t L_t = -1. \tag{2.9}$$

The Darboux linearization is given by

$$z_1 = H_1(z,w) = f_0 f_1^{\alpha_1} \cdots f_s^{\alpha_s}, \qquad y_1 = H_2(z,w) = g_0 g_1^{\beta_1} \cdots g_t^{\beta_t}.$$

Sometimes we cannot find enough Darboux factors to construct Darboux linearizations of both equations of the system. Let us say that we can find only transformation z_1, which linearizes the first equation of (2.8). If we can find a first integral of system (2.8) of the form $\Psi = xy + h.o.t.$ then the second equation of (2.8) can be linearized by the transformation $w_1 = \dfrac{\Psi}{z_1}$. We note also that if system (2.8) has p irreducible Darboux factors $f_1, ..., f_p$ with associated cofactors $K_1, ..., K_p$, satisfying $s_1 K_1 + ... + s_p K_p = 0$, then the function $H = f_1^{s_1} ... f_p^{s_p}$ is a first integral of (2.8).

3. Linearizability of system (1.2)

In this section we obtain conditions for linearizability of system (1.2) with complex parameters.

Without loss of generality, we suppose that $b_{02} = -b_{20}$ in system (1.2). Indeed, if $a_{02} + a_{20} \neq 0$, we can apply the transformation $\tilde{x} = x + (b_{02} + b_{20})y/(a_{02} + a_{20})$ and $\tilde{y} = y - (b_{02} + b_{20})x/(a_{02} + a_{20})$ obtaining a system of such form, and if $a_{02} + a_{20} = 0$, we only need to make the change $(x, y) \to (y, x)$ together with the time scaling $dt = -d\tau$ to obtain the same effect.

The following theorem gives the conditions for linearizability of system (1.2).

Theorem 3.1. *Complex system (1.2) with $b_{02} = -b_{20}$ is linearizable at the origin if one of the following conditions holds:*

(1) $4a_{20}^2 + a_{11}^2 + 4a_{11}b_{20} + 4b_{20}^2 - 4a_{20}b_{11} + b_{11}^2 = r_{20} + r_{02} = a_{02} + a_{20} = 0$,
(2) $a_{02} = r_{02} = a_{11} + 2b_{20} = b_{11} - 4a_{20} = r_{11} + b_{20}^2 = r_{20} - a_{20}b_{20} = 0$,
(3) $4a_{02} + a_{20} = a_{11} + 2b_{20} = 2b_{11} - a_{20} = 4r_{02} + a_{20}b_{20} = r_{11} + b_{20}^2 = r_{20} - a_{20}b_{20} = 0$,
(4) $a_{02} = r_{02} = a_{11} + 2b_{20} = b_{11} - a_{20} = r_{20} - a_{20}b_{20} = 0$,
(5) $9a_{11}^2 - 12a_{11}b_{20} + 4b_{20}^2 + 4b_{11}^2 = -6a_{11}b_{20} + 4b_{20}^2 + 2a_{20}b_{11} - b_{11}^2 = 6a_{20}a_{11} - 4a_{20}b_{20} - 3a_{11}b_{11} + 10b_{20}b_{11} = 4a_{20}^2 - 12a_{11}b_{20} + 24b_{20}^2 - b_{11}^2 = -\frac{4}{3}b_{20}^2 - \frac{b_{11}^2}{3} + r_{11} = \frac{4}{9}a_{20}b_{20} + \frac{a_{11}b_{11}}{6} - \frac{b_{20}b_{11}}{9} + r_{02} = \frac{a_{20}a_{11}}{6} - \frac{a_{20}b_{20}}{3} + \frac{a_{11}b_{11}}{12} - \frac{b_{20}b_{11}}{6} + r_{20} + r_{02} = a_{02} + \frac{a_{20}}{3} - \frac{b_{11}}{3} = 0$.

Proof. Using the computer algebra system Mathematica following the computational procedure described in the previous section we computed the first eight pairs of the linearizability quantities for system (1.2). The first pair is

$$i_1 = \frac{1}{9}(10a_{02}^2 + a_{11}^2 + 10a_{02}a_{20} + 4a_{20}^2 - a_{02}b_{11} - 5a_{20}b_{11} + b_{11}^2 + 4a_{11}b_{20} + 4b_{20}^2),$$
$$j_1 = \frac{1}{3}(a_{02}a_{11} + a_{11}a_{20} - 2a_{02}b_{20} - 2a_{20}b_{20} + 4r_{02} + 4r_{20}),$$

and the second pair reduced by the Groebner basis of $\langle i_1, j_1\rangle$ is

$$\begin{aligned}\tilde{i}_2 = \frac{1}{750}(&-10a_{11}^2a_{20}^2 + 200a02a_{20}^3 + 160a_{20}^4 + 10a_{11}^2a_{20}b_{11} - 600a_{02}a_{20}^2b_{11} - 520a_{20}^3b_{11} \\ &+ 6a_{11}^2b_{11}^2 + 490a_{02}a_{20}b_{11}^2 + 464a_{20}^2b_{11}^2 - 96a_{02}b_{11}^3 - 110a_{20}b_{11}^3 + 6b_{11}^4 - 170a_{11}^3b_{20} \\ &- 720a_{11}a_{20}^2b_{20} + 720a_{11}a_{20}b_{11}b_{20} - 146a_{11}b_{11}^2b_{20} - 550a_{11}^2b_{20}^2 + 2600a_{02}a_{20}b_{20}^2 \\ &+ 3080a_{20}^2b_{20}^2 - 1580a_{02}b_{11}b_{20}^2 - 2060a_{20}b_{11}b_{20}^2 + 154b_{11}^2b_{20}^2 - 160a_{11}b_{20}^3 + 520b_{20}^4 \\ &- 100a_{11}a_{20}r_{02} + 560a_{11}b_{11}r_{02} - 6800a_{20}b_{20}r_{02} + 2180b_{11}b_{20}r_{02} + 5250r_{02}^2 \\ &- 55a_{11}^2r_{11} + 50a_{02}a_{20}r_{11} - 170a_{20}^2r_{11} + 5a_{02}b_{11}r_{11} + 225a_{20}b_{11}r_{11} - 55b_{11}^2r_{11} \\ &- 220a_{11}b_{20}r_{11} - 220b_{20}^2r_{11} - 100a_{11}a_{20}r_{20} + 560a_{11}b_{11}r_{20} + 800a_{02}b_{20}r_{20} \\ &- 6000a_{20}b_{20}r_{20} + 2180b_{11}b_{20}r_{20} + 8500r_{02}r_{20} + 3250r_{20}^2),\end{aligned}$$

$$\begin{aligned}\tilde{j}_2 = \frac{1}{120}(&2a_{11}^3a_{20} + 8a_{11}a_{20}^3 - a_{11}^3b_{11} - 12a_{11}a_{20}^2b_{11} + 6a_{11}a_{20}b_{11}^2 - a_{11}b_{11}^3 - 4a_{11}^2a_{20}b_{20} \\ &+ 48a_{02}a_{20}^2b_{20} - 6a_{11}^2b_{11}b_{20} + 16a_{02}a_{20}b_{11}b_{20} + 56a_{20}^2b_{11}b_{20} - 4a_{02}b_{11}^2b_{20} \\ &- 8a_{20}b_{11}^2b_{20} - 2b_{11}^3b_{20} - 40a_{11}a_{20}b_{20}^2 - 12a_{11}b_{11}b_{20}^2 - 48a_{20}b_{20}^3 - 8b_{11}b_{20}^3 \\ &- 24a_{11}^2r_{02} - 16a_{20}^2r_{02} + 64a_{20}b_{11}r_{02} + 4b_{11}^2r_{02} - 64a_{11}b_{20}r_{02} - 32b_{20}^2r_{02} \\ &+ 128a_{02}b_{20}r_{11} + 128a_{20}b_{20}r_{11} - 128r_{02}r_{11} - 8a_{11}^2r_{20} - 32a_{02}a_{20}r_{20} + 16a_{20}^2r_{20} \\ &- 80a_{02}b_{11}r_{20} - 80a_{20}b_{11}r_{20} + 20b_{11}^2r_{20} + 32b_{20}^2r_{20} - 128r_{11}r_{20}).\end{aligned}$$

The other polynomials have very long expressions, so we do not present them here, however, the reader can easily compute them using any available computer algebra system.

To find conditions for linearizability we have to "solve" the system $i_1 = \cdots = i_8 = j_1 = \cdots = j_8 = 0$, or, more precisely, to find the irreducible decomposition of the variety $\mathbf{V}(\mathcal{L}_8)$ of the ideal $\mathcal{L}_8 = \langle i_1, j_1, ..., i_8, j_8 \rangle$. Although nowadays there are few algorithms for computing such decompositions the calculations seldom can be completed over the field of rational numbers for non-trivial ideals due to high complexity of Groebner bases computations. We tried to perform the decomposition of the variety of $\mathbf{V}(\mathcal{L}_8)$ using the routine `minAssGTZ` [14] of Singular [13], however we have not succeeded to complete computations neither over $\mathbb{Q}$ nor over the field $\mathbb{Z}_{32003}$.

To find the decomposition we proceed as follows. First, we use the conditions for isochronicity of real system (1.2) obtained in [22], which are conditions (2)–(4) of the statement of the theorem and the condition

$$a_{02} + a_{20} = a_{11} + 2b_{20} = b_{11} - 2a_{20} = r_{02} + r_{20} = 0. \tag{3.1}$$

It is clear that under condition (3.1) and conditions (2)–(4) of the theorem complex system (1.2) is linearizable as well.

Denote by J_1 the ideal generated by polynomials defining condition (3.1), that is,

$$J_1 = \langle a_{02} + a_{20}, a_{11} + 2b_{20}, b_{11} - 2a_{20}, r_{02} + r_{20} \rangle,$$

and by J_2, J_3, J_4 ideals generated by polynomials of conditions (2)–(4) of the theorem.

As we have mentioned above we are not able to compute the decomposition of the variety $\mathbf{V}(\mathcal{L}_8)$ of $\mathcal{L}_8$ (that is, to find the minimal associate primes of $\mathcal{L}_8$) even over fields of finite characteristic. However using the ideals J_1–J_4 we can find the decomposition of the variety $\mathbf{V}(\mathcal{L}_8)$. The idea is to subtract from $\mathbf{V}(\mathcal{L}_8)$ the components defined by the ideals J_1–J_4 and then find the decomposition of the remaining variety. For this aim we use the theorem (see, e.g. [12, Chapter 4] for the proof), which says that given two ideals I and H in $k[x_1, \ldots, x_n]$,

$$\overline{\mathbf{V}(I) \setminus \mathbf{V}(H)} \subset \mathbf{V}(I : H)\,,$$

where the overline indicates the Zariski closure. Moreover, if $k = \mathbb{C}$ and I is a radical ideal, then

$$\overline{\mathbf{V}(I) \setminus \mathbf{V}(H)} = \mathbf{V}(I : H). \tag{3.2}$$

Thus, to remove the components $\mathbf{V}(J_1), ..., \mathbf{V}(J_4)$ from $\mathbf{V}(\mathcal{L}_8)$, we compute over the field $\mathbb{Z}_{32003}$ with the `intersect` of Singular the intersection $J = J_1 \cap J_2 \cap J_3 \cap J_4$ (clearly, $\mathbf{V}(J) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) \cup \mathbf{V}(J_3) \cup \mathbf{V}(J_4)$), then with the `radical` we compute $R = \sqrt{\mathcal{L}_8}$, then with `quotient` we compute the ideal $G = R : J$ and, finally, with `minAssGTZ` we compute the minimal associate primes of G, obtaining that $G = G_1 \cap G_2$, where $G_1 = \langle r_{20} + r_{02}, a_{02} + a_{20}a_{20}^2 + 8001a_{11}^2 + a_{11}b_{20} + b_{20}^2 - a_{20}b_{11} + 8001b_{11}^2 \rangle$ and $G_2 = \langle a_{02} + 10668a_{20} - 10668b_{11}, a_{11}r_{02} - 10667b_{20}r_{02} + 14224a_{20}r_{11} - 14224b_{11}r_{11}, a_{20}r_{02} + 16000b_{11}r_{02} + 16001a_{11}r_{11} - b_{20}r_{11}, r_{20}^2 + 6r_{20}r_{02} + 9r_{02}^2 + r_{11}^2, b_{11}r_{20} + 3b_{11}r_{02} - 16000a_{11}r_{11} - b_{20}r_{11}, b_{20}r_{20} + 3b_{20}r_{02} - 16001a_{20}r_{11} - 8001b_{11}r_{11}, a_{11}r_{20} - 2b_{20}r_{02} - a_{20}r_{11} - 16001b_{11}r_{11}, a_{20}r_{20} - 15997b_{11}r_{02} + 8003a_{11}r_{11} - 16001b_{20}r_{11}, a_{11}b_{11} - 2b_{20}b_{11} + 4r_{20} + 6r_{02}, b_{20}^2 + 8001b_{11}^2 + 8000r_{11}, a_{11}b_{20} - 10668a_{20}b_{11} + 10668b_{11}^2 + 16001r_{11}, a_{20}b_{20} - 16001b_{20}b_{11} + 16000r_{20}, a_{11}^2 - 14224a_{20}b_{11} - 7111b_{11}^2 - 10668r_{11}, a_{20}a_{11} + b_{20}b_{11} + r_{20} + 3r_{02}, a_{20}^2 - a_{20}b_{11} + 8000b_{11}^2 + 3r_{11}, a_{20}b_{11}r_{11} - 16001b_{11}^2r_{11} - 9r_{20}r_{02} - 27r_{02}^2 - 6r_{11}^2, b_{20}b_{11}r_{02} - 8001b_{11}^2r_{11} + 8003r_{02}^2 + r_{11}^2, a_{20}b_{11}^2 - 16001b_{11}^3 - 18b_{20}r_{02} - 6a_{20}r_{11}, b_{11}^2r_{02}r_{11} + b_{20}b_{11}r_{11}^2 + 15997r_{20}r_{02}^2 + 15988r_{02}^3 - r_{20}r_{11}^2 + 15997r_{02}r_{11}^2, b_{11}^3r_{02} + b_{20}b_{11}^2r_{11} - 9b_{20}r_{02}^2 - 3b_{11}r_{02}r_{11} - 4b_{20}r_{11}^2 \rangle$. Since $8001 \equiv \frac{1}{4} \bmod 32003$, lifting the ideal G_1 from the ring of polynomials over the field $\mathbb{Z}_{32003}$ to the ring of polynomials over the field $\mathbb{Q}$ we obtain polynomials given in condition (1) of the theorem.

Similarly, lifting the ideal G_2 we obtain the ideal which we denote by J_5 (the lifting can be performed algorithmically using the algorithm of [29] and the Mathematica code of [16]). Simple computations show that $\mathbf{V}(J_5)$ is the same set as the set given by conditions (5) of the theorem.

To check the correctness of the obtained conditions we use the procedure described in [27]. First, we computed the ideal $\tilde{J} = J_1 \cap J_2 \cap J_3 \cap J_4 \cap J_5$, which defines the union of all five components of the theorem and checked that Groebner bases of all ideals $\langle \tilde{J}, 1 - wi_k \rangle$, $\langle \tilde{J}, 1 - wj_k \rangle$ (where $k = 1, \dots, 8$ and w is a new variable) computed over $\mathbb{Q}$ are $\{1\}$. By the Radical Membership Test (see e.g. [12,28]) it means that

$$\mathbf{V}(\mathcal{L}_8) \subset \mathbf{V}(\tilde{J}).$$

To check the opposite inclusion it is sufficient to check that

$$\langle \mathcal{L}_8, 1 - wf \rangle = \langle 1 \rangle \tag{3.3}$$

for all polynomials f from a basis of $\tilde{J}$. Unfortunately, we were not able to perform the check over $\mathbb{Q}$, however we have checked that (3.3) holds over few fields of finite characteristic. It yields that (3.3) holds with high probability [2].[1]

We now prove that under each of conditions (1)–(5) of the theorem the system is linearizable.

Case (1): In this case $a_{11} = -2b_{20} \pm (2a_{20} - b_{11})i$. We consider only the case $a_{11} = -2b_{20} + (2a_{20} - b_{11})i$, since when $a_{11} = -2b_{20} - (2a_{20} - b_{11})i$ the consideration is analogous. In this case system (1.2) becomes

$$\begin{aligned} \dot{x} &= -y + a_{20}x^2 + (-2b_{20} + (2a_{20} - b_{11})i)xy - a_{20}y^2 + r_{20}x^3 + r_{11}x^2y - r_{20}xy^2, \\ \dot{y} &= x + b_{20}x^2 + b_{11}xy - b_{20}y^2 + r_{20}x^2y + r_{11}xy^2 - r_{20}y^3. \end{aligned} \tag{3.4}$$

After the substitution (2.7) we obtain from (3.4) the system

$$\begin{aligned} \dot{z} &= z - (ia_{20} - b_{20})z^2 - \frac{1}{4}(r_{11} + 2ir_{20})z^3 + \frac{1}{4}(r_{11} - 2ir_{20})zw^2, \\ \dot{w} &= -w + \frac{1}{2}(ib_{11} - 2ia_{20})z^2 - \frac{1}{2}(ib_{11} + 2b_{20})w^2 - \frac{1}{4}(r_{11} + 2ir_{20})z^2w + \frac{1}{4}(r_{11} - 2ir_{20})w^3. \end{aligned} \tag{3.5}$$

System (3.5) has Darboux factors

$$\begin{aligned} l_1 &= z, \\ l_3 &= 1 + \frac{1}{16}(-8ia_{20} + 8b_{20} - 4\sqrt{2}\eta_-)z + \frac{1}{4}(ib_{11} + 2b_{20} - i\xi)w, \\ l_4 &= 1 + \frac{1}{16}(-8ia_{20} + 8b_{20} + 4\sqrt{2}\eta_-)z + \frac{1}{4}(ib_{11} + 2b_{20} - i\xi)w, \\ l_5 &= 1 + \frac{1}{16}(-8ia_{20} + 8b_{20} - 4\sqrt{2}\eta_+)z + \frac{1}{4}(ib_{11} + 2b_{20} + i\xi)w, \\ l_6 &= 1 + \frac{1}{16}(-8ia_{20} + 8b_{20} + 4\sqrt{2}\eta_+)z + \frac{1}{4}(ib_{11} + 2b_{20} + i\xi)w, \end{aligned}$$

where $\xi = \sqrt{b_{11}^2 - 4ib_{11}b_{20} - 4b_{20}^2 - 4r_{11} + 8ir_{20}}$ and

$$\eta_\pm = \sqrt{-2a_{20}^2 + 2a_{20}b_{11} - b_{11}^2 - 8ia_{20}b_{20} + 2ib_{11}b_{20} + 2b_{20}^2 + 2r_{11} + 4ir_{20} \pm 2a_{20}\xi \mp b_{11}\xi}.$$

[1] For this reason we say in the statement of Theorem 3.1 that conditions (1)–(5) are only necessary, but not necessary and sufficient conditions for linearizability of system (1.2).

It is easy to verify that the first of conditions (2.9) is satisfied with $f_0 = l_1$, $f_1 = l_4$, $f_2 = l_5$, $f_3 = l_6$, and

$$\alpha_1 = -\frac{b_{11} - 2ib_{20} + \xi}{2\xi},$$
$$\alpha_2 = \frac{b_{11}\eta_+ - 2ib_{20}\eta_+ - b_{11}\eta_- + 2ib_{20}\eta_- - 2i\sqrt{2}a_{20}\xi + 2\sqrt{2}b_{20}\xi - \eta_+\xi - \eta_-\xi}{4\eta_+\xi},$$
$$\alpha_3 = \frac{b_{11}(\eta_+ + \eta_-) + (2i\sqrt{2}a_{20} - \eta_+ + \eta_-)\xi - 2ib_{20}(\eta_+ + \eta_- - i\sqrt{2}\xi)}{4\eta_+\xi}.$$

Moreover, the system has the Darboux first integral

$$\Psi(z, w) = l_3^{s_1} l_4^{s_2} l_5^{s_3} l_6^{s_4} = 1 - \frac{i}{2\sqrt{2}}\eta_-\xi zw + o(||(z, w)||^3),$$

where $s_1 = 1$, $s_2 = -1$, $s_3 = -\frac{\eta_-}{\eta_+}$, $s_4 = \frac{\eta_-}{\eta_+}$, $f_1 = l_3$, $f_2 = l_4$, $f_3 = l_5$, and $f_4 = l_6$.

Therefore, the system is linearizable by the substitution

$$z_1 = l_1 l_4^{\alpha_1} l_5^{\alpha_2} l_6^{\alpha_3}, \quad w_1 = \frac{2\sqrt{2}(\Psi(z, w) - 1)i}{\eta_-\xi z_1}.$$

Case (2): In this case system (1.2) becomes

$$\begin{aligned} \dot{x} &= -y + a_{20}x^2 + a_{20}b_{20}x^3 - 2b_{20}xy - b_{20}^2x^2y = (b_{20}x + 1)(a_{20}x^2 - b_{20}xy - y), \\ \dot{y} &= x + b_{20}x^2 + 4a_{20}xy + a_{20}b_{20}x^2y - b_{20}y^2 - b_{20}^2xy^2, \end{aligned} \tag{3.6}$$

and after substitution (2.7) we have the system

$$\begin{aligned} \dot{z} &= z + \left(b_{20} - i\frac{5}{4}a_{20}\right)z^2 - \frac{i}{2}a_{20}zw + i\frac{3}{4}a_{20}w^2 + \left(\frac{b_{20}^2}{4} - \frac{i}{4}a_{20}b_{20}\right)z^3 \\ &\quad - \frac{i}{2}a_{20}b_{20}z^2w - \left(\frac{b_{20}^2}{4} + \frac{i}{4}a_{20}b_{20}\right)zw^2, \\ \dot{w} &= -w + i\frac{3}{4}a_{20}z^2 - \frac{i}{2}a_{20}zw - \left(b_{20} + i\frac{5}{4}a_{20}\right)w^2 + \left(\frac{b_{20}^2}{4} - \frac{i}{4}a_{20}b_{20}\right)z^2w \\ &\quad - \frac{i}{2}a_{20}b_{20}zw^2 - \left(\frac{b_{20}^2}{4} + \frac{i}{4}a_{20}b_{20}\right)w^3. \end{aligned} \tag{3.7}$$

System (3.7) has the Darboux factors

$$\begin{aligned} l_1 &= z + \left(\frac{b_{20}}{2} + \frac{i}{4}a_{20}\right)z^2 + \left(\frac{b_{20}}{2} + \frac{i}{2}a_{20}\right)zw + \frac{i}{4}a_{20}w^2, \\ l_2 &= w - \frac{i}{4}a_{20}z^2 + \left(\frac{b_{20}}{2} - \frac{i}{2}a_{20}\right)zw + \left(\frac{b_{20}}{2} - \frac{i}{4}a_{20}\right)w^2, \\ l_3 &= 1 + \frac{b_{20}}{2}z + \frac{b_{20}}{2}w, \\ l_4 &= 1 - \frac{i}{2}(4a_{20} + ib_{20})z + \frac{1}{2}(b_{20} + 4ia_{20})w, \end{aligned}$$

which, when $a_{20} \neq 0$, allow to construct the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where

$$\alpha_1 = -\frac{6a_{20} - ib_{20}}{4a_{20}}, \quad \alpha_2 = -\frac{2a_{20} + ib_{20}}{4a_{20}},$$
$$\beta_1 = -\frac{6a_{20} + ib_{20}}{4a_{20}}, \quad \beta_2 = -\frac{2a_{20} - ib_{20}}{4a_{20}}.$$

Since the set of linearizable system is an affine variety and therefore it is a closed set in the Zariski topology, the system is linearizable also when $a_{20} = 0$.

Case (3): In this case system (1.2) becomes

$$\begin{aligned} \dot{x} &= -y + a_{20}x^2 - 2b_{20}xy - \frac{a_{20}}{4}y^2 + x\left(a_{20}b_{20}x^2 - b_{20}^2xy - \frac{a_{20}b_{20}}{4}y^2\right), \\ \dot{y} &= x + b_{20}x^2 + \frac{a_{20}}{2}xy - b_{20}y^2 + y\left(a_{20}b_{20}x^2 - b_{20}^2xy - \frac{a_{20}b_{20}}{4}y^2\right), \end{aligned} \tag{3.8}$$

and the corresponding system of the form (2.8) is

$$\begin{aligned} \dot{z} &= z + \left(b_{20} - i\frac{7}{16}a_{20}\right)z^2 - i\frac{3}{8}a_{20}zw - i\frac{3}{16}a_{20}w^2 + \left(\frac{b_{20}^2}{4} - i\frac{5}{16}a_{20}b_{20}\right)z^3 \\ &\quad - i\frac{3}{8}a_{20}b_{20}z^2w - \left(\frac{b_{20}^2}{4} + i\frac{5}{16}a_{20}b_{20}\right)zw^2, \\ \dot{w} &= -w - i\frac{3}{16}a_{20}z^2 - i\frac{3}{8}a_{20}zw - \left(b_{20} + i\frac{7}{16}a_{20}\right)w^2 + \left(\frac{b_{20}^2}{4} - i\frac{5}{16}a_{20}b_{20}\right)z^2w \\ &\quad - i\frac{3}{8}a_{20}b_{20}zw^2 - \left(\frac{b_{20}^2}{4} + i\frac{5}{16}a_{20}b_{20}\right)w^3. \end{aligned} \tag{3.9}$$

System (3.9) has the following Darboux factors

$$\begin{aligned} l_1 &= z + \left(\frac{b_{20}}{2} - \frac{i}{16}a_{20}\right)z^2 + \left(\frac{b_{20}}{2} + \frac{i}{8}a_{20}\right)zw - \frac{i}{16}a_{20}w^2, \\ l_2 &= w + \frac{i}{16}a_{20}z^2 + \left(\frac{b_{20}}{2} - \frac{i}{8}a_{20}\right)zw + \left(\frac{b_{20}}{2} + \frac{i}{16}a_{20}\right)w^2, \\ l_3 &= 1 + \frac{b_{20}}{2}z + \frac{b_{20}}{2}w, \\ l_4 &= 1 - \frac{i}{4}\left(a_{20} + i2b_{20}\right)z + \frac{i}{4}\left(a_{20} - i2b_{20}\right)w, \end{aligned}$$

which, when $a_{20} \neq 0$, allow to construct the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where

$$\alpha_1 = \frac{i2b_{20}}{a_{20}}, \quad \alpha_2 = -\frac{2a_{20} + i2b_{20}}{a_{20}},$$
$$\beta_1 = -\frac{i2b_{20}}{a_{20}}, \quad \beta_2 = -\frac{2a_{20} - i2b_{20}}{a_{20}}.$$

If $a_{20} = 0$, case (3) is equivalent to case (2). Thus system (3.9) is linearizable.

Case (4): In this case system (1.2) becomes

$$\begin{aligned}\dot{x} &= -y + a_{20}x^2 - 2b_{20}xy + a_{20}b_{20}x^3 + r_{11}x^2y,\\ \dot{y} &= x + b_{20}x^2 + a_{20}xy - b_{20}y^2 + a_{20}b_{20}x^2y + r_{11}xy^2,\end{aligned} \tag{3.10}$$

and after the substitution (2.7) we obtain the system

$$\begin{aligned}\dot{z} &= z + (b_{20} - i/2a_{20})\, z^2 - \frac{i}{2}a_{20}zw - \left(\frac{r_{11}}{4} + \frac{i}{4}a_{20}b_{20}\right) z^3 - \frac{i}{2}a_{20}b_{20}z^2w\\ &\quad + \left(\frac{r_{11}}{4} - \frac{i}{4}a_{20}b_{20}\right) zw^2,\\ \dot{w} &= -w - \frac{i}{2}a_{20}zw - \left(b_{20} + \frac{i}{2}a_{20}\right) w^2 - \left(\frac{r_{11}}{4} + \frac{i}{4}a_{20}b_{20}\right) z^2w - \frac{i}{2}a_{20}b_{20}zw^2\\ &\quad + \left(\frac{r_{11}}{4} - \frac{i}{4}a_{20}b_{20}\right) w^3,\end{aligned}$$

which admits the Darboux factors

$$\begin{aligned}l_1 &= z, \quad l_2 = w,\\ l_3 &= 1 + \frac{1}{4}\left(-ia_{20} + 2b_{20} + iC\right) z - \frac{i}{4}\left(-a_{20} + i2b_{20} + C\right) w,\\ l_4 &= 1 - \frac{i}{2}\left(a_{20} + i2b_{20}\right) z + \frac{i}{2}\left(a_{20} - i2ib_{20}\right) w - \frac{i}{4}\left(a_{20}b_{20} - ir_{11}\right) z^2\\ &\quad + \frac{1}{2}\left(2b_{20}^2 + r_{11}\right) zw + \frac{i}{4}\left(a_{20}b_{20} + ir_{11}\right) w^2,\end{aligned}$$

where $C = \sqrt{a_{20}^2 - 4b_{20}^2 - 4r_{11}}$. When $C \neq 0$ we obtain the Darboux linearization

$$z_1 = l_1 l_3^{\alpha_1} l_4^{\alpha_2}, \quad w_1 = l_2 l_3^{\beta_1} l_4^{\beta_2},$$

where

$$\begin{aligned}\alpha_1 &= \frac{a_{20} + i2b_{20}}{C}, \quad \alpha_2 = -\frac{a_{20} + i2b_{20} + C}{2C},\\ \beta_1 &= \frac{a_{20} - i2b_{20}}{C}, \quad \beta_2 = -\frac{a_{20} - i2b_{20} + C}{2C}.\end{aligned}$$

Using the same argument as in case (2) we conclude that the system is linearizable also when $C = 0$.

Case (5): If $b_{20} \neq 0$, we can rewrite the condition as

$$\begin{aligned}r_{11} &= 3a_{02}^2 + 2a_{20}a_{02} + \frac{a_{20}^2}{3} + \frac{4b_{20}^2}{3}, \quad r_{02} = \frac{27a_{02}^3 + 9a_{02}^2a_{20} - 3a_{02}a_{20}^2 - a_{20}^3 - 16a_{20}b_{20}^2}{36b_{20}},\\ a_{11} &= -\frac{9a_{02}^2 - a_{20}^2 - 4b_{20}^2}{6b_{20}}, \; r_{20} = a_{02}b_{20} + a_{20}b_{20}, \; b_{11} = a_{20} + 3a_{02}, \; a_{20} = 3a_{02} \pm 4b_{20}i.\end{aligned}$$

We only consider the case $a_{20} = 3a_{02} + 4b_{20}i$, since when $a_{20} = 3a_{02} - 4b_{20}i$, the consideration is analogous. Under this condition after the substitution (2.7) system (1.2) becomes

$$\begin{aligned}\dot{z} &= z + (3b_{20} - 3ia_{02})z^2 + (2b_{20} - 2ia_{02})zw + 2ia_{02}w^2 + (2b_{20}^2 - 2a_{02}^2 - 4ia_{02}b_{20})z^3 \\ &\quad - (2a_{02}^2 + 4ia_{02}b_{20} - 2b_{20}^2)z^2w + (4a_{02}^2 + 4ia_{02}b_{20})zw^2, \\ \dot{w} &= -w\Big(1 + (2ia_{02} - 2b_{20})z + (ia_{02} - b_{20})w + (2a_{02}^2 + 4ia_{02}b_{20} - 2b_{20}^2)z^2 \\ &\quad + (2a_{02}^2 + 4ia_{02}b_{20} - 2b_{20}^2)zw - (4a_{02}^2 + 4ia_{02}b_{20})w^2\Big).\end{aligned} \tag{3.11}$$

System (3.11) has the Darboux factors

$$\begin{aligned} l_1 &= z - i(a_{02} + ib_{20})z^2 + \frac{2ia_{02}}{3}w^2, \\ l_2 &= w, \\ l_3 &= 1 - 2i(a_{02} + ib_{20})z + i(a_{02} + ib_{20})w, \\ l_4 &= 1 - 4i(a_{02} + ib_{20})z - 4(a_{02} + ib_{20})^2z^2 + 2i(a_{02} + ib_{20})w + 4(a_{02} + ib_{20})^2zw \\ &\quad + (-a_{02}^2 - 2ia_{02}b_{20} + b_{20}^2)w^2, \end{aligned}$$

which allow to construct the Darboux linearization

$$z_1 = l_1l_4^{-1}, \quad w_1 = l_2l_4^{-\frac{1}{2}}.$$

Similarly as above, using the Zariski closure argument we conclude that the system is linearizable also when $b_{20} = 0$. □

4. Relation between isochronicity conditions of [5] and Theorem 3.1

In [5] the authors presented conditions for isochronicity of system (1.2) when all parameters of the system are real. We investigate the relation between their conditions and the conditions presented in Theorem 3.1 and in [22]. The following result is obtained in [5].

Theorem 4.1 *(Theorem 1 of [5]). The origin of system (1.2) is an isochronous center if and only if (1.2) can be transformed in one of the following forms in polar coordinates:*

(a) $\dot{r} = r^2(\cos 3\theta - \frac{7}{3}\cos\theta - k_1\sin\theta) + r^3(-\frac{2k_1}{3} - \frac{2k_1}{3}\cos 2\theta - \frac{k_1^2}{2}\sin 2\theta)$, $\dot{\theta} = 1 + r(-\sin 3\theta + k_1\cos\theta - \sin\theta)$,
(b) $\dot{r} = r^2(\cos 3\theta + \frac{13}{3}\cos\theta - k_1\sin\theta) + r^3(2k_1 + \frac{10k_1}{3}\cos 2\theta - \frac{k_1^2}{2}\sin 2\theta)$, $\dot{\theta} = 1 + r(-\sin 3\theta + k_1\cos\theta + \frac{1}{3}\sin\theta)$,
(c) $\dot{r} = r^2k_1\cos\theta + r^3(k_2\cos 2\theta + k_3\sin 2\theta)$, $\dot{\theta} = 1 + rk_1\sin\theta$,
(d) $\dot{r} = r^2(k_1\cos\theta + k_2\sin\theta) + r^3(\frac{k_1k_2}{2} - \frac{k_1k_2}{2}\cos 2\theta + k_3\sin 2\theta)$, $\dot{\theta} = 1 + rk_1\sin\theta$ *and*
(e) $\dot{r} = r^2(k_1\cos\theta + k_2\sin\theta) + r^3(k_3 + k_4\cos 2\theta + k_5\sin 2\theta)$, $\dot{\theta} = 1$,

where k_j's in each system are independent and are functions of original parameters in system (1.2).

In [5] the second equation of (c) is written as $\dot{\theta} = 1 + rk_1\cos\theta$, however it is a misprint which was corrected in [8].

As it is mentioned in the previous section by the result of [22] real system (1.2) is linearizable (equivalently, it has isochronous center) if and only if condition (3.1) or one of conditions (2)–(4) of Theorem 3.1 holds. The following theorem gives the relation of the results of [22] (and Theorem 3.1) and [5].

Theorem 4.2. *System (1.2) under conditions (3.1), (2), (3), and (4) of Theorem 3.1 can be changed into system (c), (a), (b) and (d) of Theorem 4.1, respectively.*

Proof. System (1.2) under condition (3.1) becomes

$$\begin{aligned}\dot{x} &= -y + a_{20}x^2 - 2b_{20}xy - a_{20}y^2 + x(r_{20}x^2 + r_{11}xy - r_{20}y^2) = P_1(x,y),\\ \dot{y} &= x + b_{20}x^2 + 2a_{20}xy - b_{20}y^2 + y(r_{20}x^2 + r_{11}xy - r_{20}y^2) = Q_1(x,y).\end{aligned} \tag{4.1}$$

Applying the linear transformation

$$x = -a_{20}\tilde{x} + b_{20}\tilde{y}, \quad y = b_{20}\tilde{x} + a_{20}\tilde{y}$$

and a time scaling $dt = -d\tilde{t}$, we change system (4.1) to

$$\begin{aligned}\dot{x} &= -y + k_1(x^2 - y^2) + x\left(k_2x^2 + 2k_3xy - k_2y^2\right),\\ \dot{y} &= x + 2k_1xy + y\left(k_2x^2 + 2k_3xy - k_2y^2\right),\end{aligned} \tag{4.2}$$

where $k_1 = a_{20}^2 + b_{20}^2$, $k_2 = a_{20}b_{20}r_{11} - a_{20}^2r_{20} + b_{20}^2r_{20}$, $k_3 = (a_{20}^2r_{11} - b_{20}^2r_{11} + 4a_{20}b_{20}r_{20})/2$, and below we write x and y instead of $\tilde{x}$ and $\tilde{y}$. System (4.2) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ becomes system (c).

System (1.2) under condition (2) of Theorem 3.1 becomes system (3.6). The transformation $x = \frac{4}{3a_{20}}\tilde{x}$, $y = -\frac{4}{3a_{20}}\tilde{y}$ and the time scaling $dt = -d\tilde{t}$ change system (3.6) to

$$\begin{aligned}\dot{x} &= -y - \frac{4}{3}x^2 - 2k_1xy - \frac{x}{3}\left(4k_1x^2 + 3k_1^2xy\right),\\ \dot{y} &= x + k_1x^2 - \frac{16}{3}xy - k_1y^2 - \frac{y}{3}\left(4k_1x^2 + 3k_1^2xy\right),\end{aligned} \tag{4.3}$$

where we write x and y instead of $\tilde{x}$ and $\tilde{y}$, and $k_1 = \frac{4b_{20}}{3a_{20}}$. System (4.3) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ becomes system (a).

System (1.2) under condition (3) becomes system (3.8). Applying the transformation $x = \frac{16}{3a_{20}}\tilde{x}$, $y = \frac{16}{3a_{20}}\tilde{y}$, we transform (3.8) to the system

$$\begin{aligned}\dot{x} &= -y - \frac{16}{3}x^2 - 2k_1xy - \frac{4}{3}y^2 + \frac{k_1}{3}x\left(16x^2 - 3k_1xy - 4y^2\right),\\ \dot{y} &= x + k_1x^2 + \frac{8}{3}xy - k_1y^2 + \frac{k_1}{3}y\left(16x^2 - 3k_1xy - 4y^2\right),\end{aligned} \tag{4.4}$$

where we write x and y instead of $\tilde{x}$ and $\tilde{y}$, and $k_1 = \frac{16b_{20}}{3a_{20}}$. System (4.4) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ becomes system (b).

System (1.2) under condition (4) becomes system (3.10). The transformation $x = \tilde{y}$, $y = \tilde{x}$ and a time scaling $dt = -d\tilde{t}$ change system (3.10) to

$$\begin{aligned}\dot{x} &= -y + k_1x^2 + k_2xy - k_1y^2 + x\left(2k_3xy + k_1k_2y^2\right),\\ \dot{y} &= x + 2k_1xy + k_2y^2 + y\left(2k_3xy + k_1k_2y^2\right),\end{aligned} \tag{4.5}$$

where $k_1 = b_{20}$, $k_2 = -a_{20}$, $k_3 = -\frac{r_{11}}{2}$, and we write x and y instead of $\tilde{x}$ and $\tilde{y}$. System (4.5) in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ becomes system (d). □

However system (e) from Theorem 4.1 does not have an isochronous center at the origin, since, generally speaking, the origin of the system is not a center, but a focus. Indeed, system (e) can be written in the Cartesian coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ as

$$\begin{aligned}\dot{x} &= -y + k_1x^2 + k_2xy + x\left((k_3+k_4)x^2 + (k_3-k_4)y^2 + 2k_5xy\right),\\ \dot{y} &= x + k_1xy + k_2y^2 + y\left((k_3+k_4)x^2 + (k_3-k_4)y^2 + 2k_5xy\right).\end{aligned} \tag{4.6}$$

We computed the first two Lyapunov quantities for system (4.6) and obtained $\eta_1 = k_3$ and $\eta_2 = 2k_1k_2k_5 + k_4(k_1^2 - k_2^2)$. Thus, the origin of system (e) is a focus, which is stable if $k_3 < 0$ or $k_3 = 0$, $\eta_2 < 0$, and unstable if $k_3 > 0$ or $k_3 = 0$, $\eta_2 > 0$. So, the necessary condition for existence of a center and an isochronous center at the origin of system (e) is $k_3 = \eta_2 = 0$.

When $k_3 = \eta_2 = 0$, by the linear transformation $x_1 = x + \frac{k_2}{k_1}y$, $y_1 = y - \frac{k_2}{k_1}x$, system (4.6) is changed into

$$\begin{aligned}\dot{x} &= -y + k_1x^2 - \tfrac{k_1k_4}{k_2}x^2y,\\ \dot{y} &= x + k_1xy - \tfrac{k_1k_4}{k_2}xy^2.\end{aligned} \tag{4.7}$$

System (4.7) is a special case of system (3.10) when $b_{20} = 0$, which is system (1.2) under condition (4) of Theorem 3.1 after adding the condition $b_{20} = 0$. Therefore, only when $k_3 = \eta_2 = 0$, the origin of system (4.6), and thus of system (e), is an isochronous center.

It appears the authors of [5] made the following mistake in their reasoning. They obtained system (e) from the condition of vanishing of two isochronous constants (computed using (2.3) and (2.4)). Then observing that the second equation of the system is $\dot{\theta} = 1$, they concluded that the system has an isochronous center at the origin. However, as we have shown, unless $k_3 = \eta_2 = 0$, the origin of the system is an isochronous focus (see e.g. [1,17] for definitions) but not a center.

We note that the conditions for isochronicity of system (1.2) are also given in the survey paper [8]. According to Theorem 14.2 of [8] system (1.2) has an isochronous center at the origin if and only if by a change of coordinates and rescaling of time it can be brought to one of systems (a), (b), (d) of Theorem 4.1 or to one of systems

$$\begin{aligned}\dot{r} &= r^2\cos\theta + r^3(k_2\cos 2\theta + k_3\sin 2\theta),\\ \dot{\theta} &= 1 + r\sin\theta\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\dot{r} &= r^3\cos 2\theta,\\ \dot{\theta} &= 1.\end{aligned} \tag{4.9}$$

However instead of (4.8) and (4.9) we can use just system (c) of the statement of Theorem 4.1, since system (4.8) is a particular case of system (c) if we set in (c) $k_1 = 1$ and system (4.9) is a particular case of system (c) if we set in (c) $k_1 = k_3 = 0$, $k_2 = 1$.

To summarize, in [22] the authors presented four necessary and sufficient conditions for isochronicity of the center at the origin of system (1.2), in the case when all parameters of system (1.2) are *real*. Their conditions are correct and coincide with condition (3.1) and conditions (2)–(4) of Theorem 3.1.

In [5] the authors presented five systems, which are systems (a)–(e) of Theorem 4.1 and stated that any *real* system with an isochronous center at the origin can be transformed to one of systems (a)–(e). However, as it is shown above, generally speaking system (e) has not a center, but an isochronous focus at the origin (see e.g. [1,17] for definitions). So system (e) should not be presented in the statements of Theorem 4.1.

The five systems presented in [8] are correct, however, as explained above, two of systems of [8] can be combined to give system (c) of Theorem 4.1. So there are only four necessary and sufficient conditions for isochronicity of the center at the origin of *real* system (1.2).

In our Theorem 3.1 we presented five conditions for *linearizability* of *complex* (1.2). In the case when the parameters of (1.2) are real our conditions coincide with those obtained in [22] since for real parameters condition (1) of our Theorem 3.1 is equivalent to condition (3.1) and condition (5) is equivalent to the condition that all parameters in (1.2) are equal to zero. Thus, our Theorem 3.1 contains all conditions for isochronicity of *real* system (1.2) obtained in [5] and [22], but additionally it gives also conditions for linearizability of *complex* system (1.2).

5. Coexistence of isochronous centers

In this section we present our study on existence of few isochronous centers in real system (1.2).

Theorem 5.1. *System (1.2) has at most two isochronous centers including the origin when all parameters are real. More precisely, under condition (3.1) and conditions (2), (3) and (4) of Theorem 3.1, system (1.2) has at most two, one, two and two isochronous centers, respectively.*

Proof. We first consider the simplest case, case (2) of Theorem 3.1. In this situation, system (1.2) has the form (3.6). From the first equation of (3.6), we know that the coordinates of equilibria must satisfy $b_{20}x + 1 = 0$ or $a_{20}x^2 - b_{20}xy - y = 0$. Substituting $y = a_{20}x^2/(1 + b_{20}x)$ into the right hand side of the second equation of (3.6) we obtain $4a_{20}^2x^2 + (b_{20}x + 1)^2 = 0$. Then, we get $x = 0$ or $x = -1/b_{20}$. On the other hand, substituting $x = -1/b_{20}$ into the right hand side of the second equation of (3.6), we have $-3a_{20}y/b_{20} = 0$. Thus, other than the origin $O : (0, 0)$ we get the equilibrium $A : (-1/b_{20}, 0)$ when $a_{20}b_{20} \neq 0$, no equilibria exist when $b_{20} = 0$ and $a_{20} \neq 0$, or the line $x = -1/b_{20}$ is filled by equilibria when $a_{20} = 0$ and $b_{20} \neq 0$.

Computing the determinant of linear matrix for system (3.6) at the equilibrium $A : (-1/b_{20}, 0)$, we find that it is equal to $-3a_{20}^2/b_{20}^2 < 0$, indicating that the equilibrium A is a saddle if it exists. Clearly, any equilibrium on the line $x = -1/b_{20}$ cannot be an isochronous center when $a_{20} = 0$. Therefore, in the case (2) of Theorem 3.1, system (1.2) has only one isochronous center at the origin.

Consider now case (3) of Theorem 3.1. In this case system (1.2) can be written as

$$\begin{aligned}
\dot{x} &= (b_{20}x + 1)(4b_{11}x^2 - b_{11}y^2 - 2b_{20}xy - 2y)/2 := P_3(x, y), \\
\dot{y} &= x + b_{20}x^2 + b_{11}xy - b_{20}y^2 - (b_{11}b_{20}/2)y^3 - b_{20}^2xy^2 + 2yb_{11}b_{20}x^2 := Q_3(x, y).
\end{aligned} \tag{5.1}$$

From the first equation of (5.1) we see that the coordinates of equilibria must satisfy $b_{20}x + 1 = 0$ or $g_3(x, y) := 4b_{11}x^2 - b_{11}y^2 - 2b_{20}xy - 2y = 0$. Substituting $x = -1/b_{20}$ into the right hand side of the second equation of (5.1), we have $-yb_{11}(b_{20}^2y^2 - 2)/b_{20} = 0$. Thus, we find three equilibria $A : (-1/b_{20}, 0)$ and $A_\pm : (-1/b_{20}, \pm\sqrt{2}/b_{20})$ if $b_{20} \neq 0$.

If we solve $g_3(x, y) = 0$ and substitute the solution into the right hand side of the second equation of (5.1) a very complicated expression arises. However, we only need to find the coordinates of centers for system (5.1) and at a center the trace of linear matrix is zero. We calculate

$$\begin{aligned}
T_3(x, y) := {}& \frac{\partial P_3}{\partial x} + \frac{\partial Q_3}{\partial y} \\
= {}& b_{20}(4b_{11}x^2 - b_{11}y^2 - 2b_{20}xy - 2y)/2 + (b_{20}x + 1)(8b_{11}x - 2b_{20}y)/2 \\
& + b_{11}x - 2b_{20}y - (3/2)b_{11}b_{20}y^2 - 2b_{20}^2xy + 2b_{11}b_{20}x^2,
\end{aligned}$$

$$
\begin{aligned}
D_3(x,y) &:= \frac{\partial P_3}{\partial x}\frac{\partial Q_3}{\partial y} - \frac{\partial P_3}{\partial y}\frac{\partial Q_3}{\partial x} \\
&= (b_{20}(4b_{11}x^2 - b_{11}y^2 - 2b_{20}xy - 2y)/2 + (b_{20}x+1))(8b_{11}x - 2b_{20}y)/2) \\
&\quad (b_{11}x - 2b_{20}y - (3/2)b_{11}b_{20}y^2 - 2b_{20}^2xy + 2b_{11}b_{20}x^2) \\
&\quad - (b_{20}x+1)(-2b_{11}y - 2b_{20}x - 2)(4b_{11}b_{20}xy - b_{20}^2y^2 + b_{11}y + 2b_{20}x + 1)/2.
\end{aligned}
$$

Computing a Groebner basis of the ideal $\langle g_3, Q_3, T_3\rangle$ we got the basis

$$\mathcal{G}_3 := \{b_{20}x^2 + x, b_{11}y^2 + 2b_{20}xy + 2y, b_{11}x\}.$$

When $b_{11} = 0$ we obtain the equilibrium $O : (0, 0)$ or the line $b_{20}x + 1$ is filled with equilibria. When $b_{11} \neq 0$, we obtain the equilibrium $B : (0, -2/b_{11})$.

Notice that all equilibria on the line $b_{20}x + 1$ are degenerate when $b_{11} = 0$, because the determinant of linear matrix at each equilibrium is zero. Thus, an equilibrium on the line $b_{20}x + 1$ cannot be isochronous centers when $b_{11} = 0$. By calculations, among all equilibria $A : (-1/b_{20}, 0)$, $A_\pm : (-1/b_{20}, \pm\sqrt{2}/b_{20})$ and $B : (0, -2/b_{11})$, only at B the trace of linear part is zero and the determinant of linear part is positive at the same time. So we only need to check the isochronicity of equilibrium $B : (0, -2/b_{11})$. Moving the equilibrium B to the origin and making the change

$$u = \sqrt{2}(-2b_{20}/b_{11}))x - \sqrt{2}y, \qquad v = \sqrt{2}x$$

together with the time scaling $dt = -d\tau$, we obtain from (5.1) the system

$$
\begin{aligned}
\dot{x} &= -y - \frac{\sqrt{2}b_{11}}{2}xy + \frac{\sqrt{2}b_{20}}{2}x^2 - \frac{\sqrt{2}b_{20}}{2}y^2 + \frac{b_{11}b_{20}}{4}x^3 - xb_{11}b_{20}y^2 + \frac{b_{20}^2}{2}x^2y, \\
\dot{y} &= x + \frac{\sqrt{2}b_{11}}{4}x^2 + \sqrt{2}b_{20}xy - \sqrt{2}b_{11}y^2 + \frac{b_{11}b_{20}}{4}x^2y + \frac{b_{20}^2}{2}xy^2 - b_{11}b_{20}y^3,
\end{aligned}
\tag{5.2}
$$

where we still write x, y instead of u, v. It is easy to show that system (5.2) is Darboux linearizable. Therefore, the system has isochronous centers at the origin and at the point $B : (0, -2/b_{11})$ if $b_{11} \neq 0$.

Now consider case (4) of Theorem 3.1. In this case system (1.2) has the form

$$
\begin{aligned}
\dot{x} &= -y + a_{20}x^2 - 2b_{20}xy + a_{20}b_{20}x^3 + r_{11}x^2y := P_4(x,y), \\
\dot{y} &= x + a_{20}xy + b_{20}x^2 - b_{20}y^2 + a_{20}b_{20}x^2y + r_{11}xy^2 := Q_4(x,y).
\end{aligned}
\tag{5.3}
$$

It is difficult to find the coordinates of equilibria of system (5.3) explicitly. However, we can calculate

$$
\begin{aligned}
T_4(x,y) &:= \frac{\partial P_4}{\partial x} + \frac{\partial Q_4}{\partial y}, \\
D_4(x,y) &:= \frac{\partial P_4}{\partial x}\frac{\partial Q_4}{\partial y} - \frac{\partial P_4}{\partial y}\frac{\partial Q_4}{\partial x}
\end{aligned}
$$

to find only coordinates of centers. Computing a Groebner basis of $\langle P_4, Q_4, T_4\rangle$ we obtained

$$
\begin{aligned}
\mathcal{G}_4 := \{&a_{20}xy + 4b_{20}x^2 + 4x,\ a_{20}y^2 + 4b_{20}xy + 4y,\ -3a_{20}^3x + 16a_{20}b_{20}^2x + 16a_{20}r_{11}x, \\
&a_{20}x^2 - 4b_{20}xy - 4y,\ -3a_{20}^2y + 16b_{20}^2y + 16r_{11}y,\ b_{20}x^3 + b_{20}xy^2 + x^2 + y^2, \\
&64b_{20}^3x^2 + 16b_{20}r_{11}x^2 - 3a_{20}^2x - 12a_{20}b_{20}y + 64b_{20}^2x + 16r_{11}x\}.
\end{aligned}
$$

Letting the first and the second polynomials in $\mathcal{G}_4$ be zeros, we get $y = -4(b_{20}x + 1)/a_{20}$ when $a_{20} \neq 0$ or $x = y = 0$. Substituting $y = -4(b_{20}x + 1)/a_{20}$ into $\mathcal{G}_4$, we have

$$\begin{aligned}&\{4(b_{20}x+1)(3a_{20}^2 - 16b_{20}^2 - 16r_{11})/a_{20}, (16 + (a_{20}^2 + 16b_{20}^2)x^2 + 32b_{20}x)/a_{20},\\ &(16 + (a_{20}^2 + 16b_{20}^2)x^2 + 32b_{20}x)(b_{20}x+1)/a_{20}^2, -a_{20}x(3a_{20}^2 - 16b_{20}^2 - 16r_{11}),\\ &(64b_{20}^3 + 16b_{20}r_{11})x^2 + (-3a_{20}^2 + 112b_{20}^2 + 16r_{11})x + 48b_{20}\}.\end{aligned} \tag{5.4}$$

Using the first polynomial in (5.4), we obtain $b_{20}x + 1 = 0$, $y = 0$ or $y = -4(b_{20}x + 1)/a_{20}, 3a_{20}^2 - 16b_{20}^2 - 16r_{11} = 0$. Substituting them in (5.4), we obtain

$$\begin{aligned}&\{(b_{20}x+1)x^2,\ a_{20}x^2,\ 4x(b_{20}x+1), -x(-64b_{20}^3x - 16b_{20}r_{11}x + 3a_{20}^2 - 64b_{20}^2 - 16r_{11}),\\ &-a_{20}x(3a_{20}^2 - 16b_{20}^2 - 16r_{11})\}\end{aligned}$$

and

$$\begin{aligned}&\{(a_{20}^2x^2 + (4b_{20}x+4)^2)/a_{20},\ (b_{20}x+1)(a_{20}^2x^2 + 16b_{20}^2x^2 + 32b_{20}x + 16)/a_{20}^2,\\ &3b_{20}(a_{20}^2x^2 + 16b_{20}^2x^2 + 32b_{20}x + 16)\},\end{aligned}$$

respectively. From the first and the second polynomials in above two sets, we see that on the line $y = -4(b_{20}x + 1)/a_{20}$ no center type equilibria exist when $a_{20} \neq 0$.

When $a_{20} = 0$, the basis $\mathcal{G}_4$ becomes $\{b_{20}x^2 + x, b_{20}xy + y, b_{20}^2y + r_{11}y\}$, and we obtain that $b_{20}x + 1 = 0$, $(b_{20}^2 + r_{11})y = 0$ or $x = 0$, $y = 0$. When $a_{20} = 0$, $b_{20}x + 1 = 0$ and $b_{20}^2 + r_{11} = 0$, the line $b_{20}x + 1 = 0$ is full of equilibria, none of which can be an isochronous center of system (5.3). Hence, we only get the unique possible center $A : (-1/b_{20}, 0)$ if $a_{20} = 0$, at which the trace of the linear matrix for system (5.3) is zero and the determinant is $r_{11}/b_{20}^2 + 1$. If $a_{20} = 0$, after moving the origin to the point $(-1/(2b_{20}), 0)$, system (5.3) is changed into

$$\begin{aligned}\dot{x} &= \frac{r_{11}}{4b_{20}^2}y - \frac{2b_{20}^2 + r_{11}}{b_{20}}xy + r_{11}x^2y,\\ \dot{y} &= -\frac{1}{4b_{20}} + b_{20}x^2 - \frac{2b_{20}^2 + r_{11}}{2b_{20}}y^2 + r_{11}xy^2,\end{aligned} \tag{5.5}$$

which is symmetric with respect to the x-axis. Moreover, equilibria $(\pm 1/(2b_{20}), 0)$ of system (5.5) correspond to equilibria A and O of system (5.3) respectively. Thus, except of the origin $O : (0, 0)$ we get another isochronous center at the equilibrium $A : (-1/b_{20}, 0)$ when $a_{20} = 0$, $r_{11}/b_{20}^2 + 1 > 0$ and $b_{20} \neq 0$. Therefore, in case (4) system (1.2) has at most two isochronous centers.

At last, we study the case when condition (3.1) is fulfilled. In this situation let the vector field of system (1.2) be $(P_1(x, y), Q_1(x, y))$, as shown in (4.1). Similarly to case (4), we only consider equilibria of center type avoiding complicated calculations of coordinates of all equilibria. We calculate

$$\begin{aligned}T_1(x, y) &:= \frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y},\\ D_1(x, y) &:= \frac{\partial P_1}{\partial x}\frac{\partial Q_1}{\partial y} - \frac{\partial P_1}{\partial y}\frac{\partial Q_1}{\partial x}\end{aligned}$$

to find coordinates of centers. The Groebner basis of $\langle P_1, Q_1, T_1\rangle$ is

$$\mathcal{G}_1 := \{a_{20}y + b_{20}x + 1, r_{11}xy + r_{20}x^2 - r_{20}y^2 + a_{20}x - b_{20}y, a_{20}r_{11}y^2 + a_{20}r_{20}xy + b_{20}r_{20}y^2 + a_{20}^2 y + b_{20}^2 y + r_{11}y + r_{20}x + a_{20}\}.$$

If $a_{20} = b_{20} = 0$, system (4.1) cannot have other centers except of the origin. Without loss of generality we suppose $b_{20} \neq 0$. If $a_{20} \neq 0$ the discussion is similar and we only need to make the change $(x, y) \to (y, x)$ with the time rescaling $dt = -d\tau$. From the first polynomial in $\mathcal{G}_1$, we get $x = -(a_{20}y+1)/b_{20}$. Substituting it into $\mathcal{G}_1$, we have

$$g_1 := a_0 + a_1 y + a_2 y^2 = 0, \tag{5.6}$$

where $a_0 = a_{20}b_{20} - r_{20}$, $a_1 = a_{20}^2 b_{20} + b_{20}^3 - 2a_{20}r_{20} + b_{20}r_{11}$ and $a_2 = -a_{20}^2 r_{20} + a_{20}b_{20}r_{11} + b_{20}^2 r_{20}$. Thus, from (5.6) we find two roots $y_\pm = (-a_1 \pm \sqrt{a_1^2 - 4a_2a_0})/(2a_2)$ and then get two equilibria $C_\pm$: $(-(a_{20}y_\pm + 1)/b_{20}, y_\pm)$ when $d_0 := a_1^2 - 4a_2a_0 > 0$ and $a_2 \neq 0$. At $C_\pm$ the trace of linear matrix for system (4.1) is zero and the determinant of that is

$$\tilde{D}_\pm := \frac{d_0(\mp(a_{20}^2 + b_{20}^2)\sqrt{d_0} - b_{20}(d_0/b_{20}^2 - b_{20}^2 r_{11} + 4a_{20}b_{20}r_{20} + a_{20}^2 r_{11} - r_{11}^2 - 4r_{20}^2))}{2b_{20}^3(a_{20}^2 r_{20} - a_{20}b_{20}r_{11} - b_{20}^2 r_{20})^2}.$$

Moreover,

$$\tilde{D}_+\tilde{D}_- = -\frac{d_0^2}{b_{20}^4(a_{20}^2 r_{20} - a_{20}b_{20}r_{11} - b_{20}^2 r_{20})^2} < 0,$$

implying that at most one of C_+ and C_- is a center. Actually, when $r_{20} = a_{20}b_{20}$, we find that the equilibrium $B : (0, -2/b_{11})$ is an isochronous center, since it is easy to show that system (4.1) is Darboux linearizable at this point. Therefore, if (3.1) holds, then system (1.2) has at most two isochronous centers. □

To conclude, we have found conditions for isochronicity and linearizability of system (1.2) and clarified conditions of isochronicity obtained by Chavarriga et al. in [5]. An important feature of our approach is the treatment of coefficients of system (1.2) as complex parameters, since this has allowed us to use formula (3.2) for finding the decomposition of the integrability variety and to use the Radical Membership Test in order to check the correctness of computations involved modular arithmetic.

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References

[1] V.V. Amel'kin, N.A. Lukashevich, A.P. Sadovskii, Nonlinear Oscillations in Second Order Systems, Belarusian State University, Minsk, 1982 (in Russian).
[2] E.A. Arnold, Modular algorithms for computing Gröbner bases, J. Symbolic Comput. 35 (2003) 403–419.
[3] J. Chavarriga, I.A. García, J. Giné, Isochronicity into a family of time-reversible cubic vector fields, Appl. Math. Comput. 121 (2001) 129–145.
[4] J. Chavarriga, J. Giné, Integrability of cubic systems with degenerate infinity, Differ. Equ. Dyn. Syst. 6 (1998) 425–438.
[5] J. Chavarriga, J. Giné, I. García, Isochronous centers of cubic systems with degenerate infinity, Differ. Equ. Dyn. Syst. 7 (1999) 221–238.

[6] J. Chavarriga, J. Giné, I.A. García, Isochronous centers of a linear center perturbed by fourth degree homogeneous polynomial, Bull. Sci. Math. 123 (1999) 77–96.
[7] J. Chavarriga, J. Giné, I.A. García, Isochronous centers of a linear center perturbed by fifth degree homogeneous polynomials, J. Comput. Appl. Math. 126 (2000) 351–368.
[8] J. Chavarriga, M. Sabatini, A survey of isochronous centers, Qual. Theory Dyn. Syst. 1 (1999) 1–70.
[9] X. Chen, W. Huang, V.G. Romanovski, W. Zhang, Linearizability conditions of a time-reversible quartic-like system, J. Math. Anal. Appl. 383 (2011) 179–189.
[10] X. Chen, V.G. Romanovski, W. Zhang, Linearizability conditions of time-reversible quartic systems having homogeneous nonlinearities, Nonlinear Anal. 69 (2008) 1525–1539.
[11] A. Cima, A. Gasull, V. Mañosa, F. Mañosas, Algebraic properties of the Liapunov and period constants, Rocky Mountain J. Math. 27 (1997) 471–501.
[12] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, Springer, New York, 1997.
[13] W. Decker, G.-M. Greuel, G. Pfister, H. Shönemann, SINGULAR 3-1-6—a computer algebra system for polynomial computations, http://www.singular.uni-kl.de, 2012.
[14] W. Decker, S. Laplagne, G. Pfister, H.A. Schonemann, SINGULAR 3-1 library for computing the prime decomposition and radical of ideals, primdec.lib, 2010.
[15] P. Gianni, B. Trager, G. Zacharias, Gröbner bases and primary decomposition of polynomials, J. Symbolic Comput. 6 (1988) 146–167.
[16] J. Giné, C. Christopher, M. Prešern, V.G. Romanovski, N.L. Shcheglova, The resonant center problem for a 2 : −3 resonant cubic Lotka–Volterra system, in: CASC 2012, Maribor, Slovenia, September 3–6, 2012, in: Lecture Notes in Comput. Sci., vol. 7442, 2012, pp. 129–142.
[17] J. Giné, M. Grau, Characterization of isochronous foci for planar analytic differential systems, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005) 985–998.
[18] M. Han, V.G. Romanovski, Isochronicity and normal forms of polynomial systems of ODEs, J. Symbolic Comput. 47 (2012) 1163–1174.
[19] M. Han, V.G. Romanovski, X. Zhang, Integrability of a family of 2-dim cubic systems with degenerate infinity, Romanian J. Phys. 61 (2016) 157–166.
[20] J. Li, Y. Lin, Normal form of planar autonomous system and periodic critical points of closed orbits, Acta Math. Sinica 34 (1991) 490–501.
[21] J. Llibre, C. Valls, Classification of the centers, their cyclicity and isochronicity for the generalized quadratic polynomial differential systems, J. Math. Anal. Appl. 357 (2009) 427–437.
[22] N.G. Lloyd, C.J. Christopher, J. Devlin, J.M. Pearson, N. Yasmin, Quadratic-like cubic systems, Differ. Equ. Dyn. Syst. 5 (1997) 329–345.
[23] W.S. Loud, Behaviour of the period of solutions of certain plane autonomous systems near centers, Contrib. Differ. Equ. 3 (1964) 21–36.
[24] P. Mardešić, C. Rousseau, B. Toni, Linearization of isochronous centers, J. Differential Equations 121 (1993) 67–108.
[25] I.I. Pleshkan, A new method of investigating the isochronicity of a system of two differential equations, Dokl. Akad. Nauk SSSR 182 (1968) 768–771; Sov. Math., Dokl. 9 (1968) 1205–1209.
[26] V.G. Romanovski, X. Chen, Z. Hu, Linearizability of linear systems perturbed by fifth degree homogeneous polynomials, J. Phys. A 40 (2007) 5905–5919.
[27] V.G. Romanovski, M. Prešern, An approach to solving systems of polynomials via modular arithmetics with applications, J. Comput. Appl. Math. 236 (2011) 196–208.
[28] V.G. Romanovski, D.S. Shafer, The Center and Cyclicity Problems: A Computational Algebra Approach, Birkhäuser, Boston, 2009.
[29] P.S. Wang, M.J.T. Guy, J.H. Davenport, P-adic reconstruction of rational numbers, SIGSAM Bull. 16 (1982) 2–3.
[30] K. Wu, Y. Zhao, Isochronicity for a class of reversible systems, J. Math. Anal. Appl. 365 (2010) 300–307.